Dynamic Portfolio Analysis : Mean-Variance Formulation and Iterative Parametric Dynamic Programming

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By

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To my parents

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摘要

多期投資組合選擇問題是在指定限期內尋求每一期的最優財富分佈. 本研究將馬歌斯(Markowitz)的單期平均值-方差伸展至多期問題上.採用平均值-方差模型於多期投資問題較之採用效益優化的好處是更加直接. 當投資者想得最大的末期財富，投資者只訂明他可承受的風險,便能以平均值-方差模型求得最優組合.

然而,多期平均值-方差模型引出動態規劃中的不可分解問題. 本論文提議一遞推參化動態規劃解答方案,包含埋藏,分解及凸化. 此方案在無論有否無風險資產的情況下，都可求得多期有效前線(efficient frontier)的解析解. 有效的多期投資組合數值求解方案亦建立給任何投資者其效益爲末期財產的平均值及方差的函數.

多期平均值-方差投資組合模型更能在動態環境下掌握風險管理的精神. 而且，解析解的組合方案使進行最優投資策略成爲一件簡單任務.

Abstract

Multi-period portfolio selection seeks an optimal allocation of wealth among a basket of securities at successive time periods along a time horizon under consideration. Markowitz's mean-variance formulation has been extended in this research to multi-period portfolio selection problems. One advantage of adopting a mean-variance formulation in multi-period portfolio selection over approaches of utility maximization is that it is straightforward for investors to specify the degree of risk they are able to sustain while they are seeking optimal portfolio policies to maximize their expected terminal wealth.

The mean-variance formulation in multi-period portfolio selection leads to an optimization problem which is nonseparable in the sense of dynamic programming. A solution framework using embedding, separation, and convexification is proposed. The derived solution scheme of iterative parametric dynamic programming enables an analytical expression for the efficient frontier for both situations where there is a riskless assest or there isn't one. An efficient numerical solution scheme is also developed for multi-period portfolio selection problems where an investor is maximizing a utility that is a function of both the expected value and the variance of the terminal wealth.

The mean-variance formulation in multi-period portfolio selection further captures the spirit of risk management in dynamic portfolio selection, while the derived

analytical portfolio policy makes the implementation of an optimal investment strategy an easy task.

Contents

Chapter 1

Introduction

1.1 Overview

The mean-variance approach by Markowitz[1] has provided a fundamental basis for risk management in portfolio selection. Empirical studies on the mean-variance approach have been carried out extensively, such as reported in Kroll, Levy and Markowitz[2], Levy and Markowitz [3], and Pulley[4, 5]. It has been shown that the mean-variance formulation is a good approximation for expected utility maximization in short holding period.

Almost all investment problems in our lives are of multiple periods in nature. The static mean-variance analysis would become less effective as it does not take into account the long-term impact of the short-sighted strategies. Portfolio selection cannot be performed in isolation at each individual time period. Any investor must balance the desire for high short-term return with the possibility of high future loss.

The papers by Mossin[6] and Samuelson[7] were the earliest discussion of multi-period portfolio analysis. Since then, there have been intermittent studies in multi-period portfolio analysis. Elton and Gruber[8], and Francis[9] gave brief summaries on early contributions in the development of dynamic portfolio management. Later, Grauer and Hakansson[10] developed a general objective function of power form which covers a large range of investors' behavior from risk neutral to risk aversion. A recent review by Dahl et al[11] gave invaluable comments on limitations of the above formulations.

Although the past two decades have witnessed many successful applications of dynamic programming in multi-period portfolio selection as a powerful solution scheme in solving various problem formulations of utility maximization in terms of the terminal wealth [6, 7, 8, 10, 12, 13, 14]. Most of the results in utility maximization in the literature has been dominated by myopic optimal policies for special forms of utility functions [8, 11, 12, 13]. One crutial condition to apply dynamic programming is that the utility function must be separable.

From investors' point of view, it is very difficult for any investor to construct an explicit utility function of the final wealth with predetermined coefficients. One easy and practical way to elicitate an investor's preference would be based on trade-offs between the expected value and the variance of the terminal wealth. It is thus recommended in this research to fully utilize the philosophy of Markowitz's approach for the dynamic portfolio selection. Optimal investment decisions should be sought by first generating the efficient frontier in the return-risk space for the whole range of risk aversion and then selecting the best-compromised portfolio for each particular investor.

The motivation of this research is to develop dynamic mean-variance models and solution schemes that extend the existing literature to capture the spirit of risk management in multi-period portfolio analysis. We consider a capital market with a basket of risky assets with random rates of return and a risk-less asset offering a sure rate of return. An investor joins the market at time 0 with an initial wealth x_0. The investor can allocate his wealth among the various securities. The current wealth can be reinvested at the beginning of each of $T - 1$ consecutive time periods. Specifically, the wealth dynamics is governed by the following stochastic difference equation,

$$x_{t+1} = s_t x_t + \sum_{i=1}^{n} \left(e_t^i - s_t\right) u_t^i \quad t = 0, 1, 2, ..., T-1 \tag{1.1}$$

where t is the index for time period, x_t is the wealth of the investor at the beginning of period t, s_t is the rate of return of the riskless asset during the t-th period, e_t^i is the random return of the i-th risky asset during the period t, and u_t^i is the amount invested at the beginning of the t-th period in the i-th risky asset. It is assumed that the mean and variance-covariance of risky returns at time t, for

$t = 0, 1, ..., T-1$, are known to be $E(e_t) = E(e_t^1, e_t^2, ..., e_t^n)'$, and $Cov(e_t) = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} & \cdots & \sigma_{1n,t} \\ \sigma_{21,t} & \sigma_{22,t} & \cdots & \sigma_{2n,t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1,t} & \sigma_{n2,t} & \cdots & \sigma_{nn,t} \end{bmatrix}$.

The objective of an investor is to maximize the expected value of the terminal wealth x_T, $E(x_T)$, while at the same time to minimize the risk level that is measured by $Var(x_T)$. An analytical expression of the efficient frontier will be derived in this research. For an investor with a specified risk level, the optimal portfolio policy can be generated via solving the following dynamic optimization problem,

$$\begin{aligned} E(w) \quad &: \quad \max \; E(x_T) - wVar(x_T) && (1.2) \\ s.t. \quad x_{t+1} &= s_t x_t + \sum_{i=1}^{n} \left(e_t^i - s_t\right) u_t^i \quad t = 0, 1, 2, ..., T-1 && (1.3) \end{aligned}$$

where w represents the degree of the risk aversion of the investor.

Investors' behavior, in a more general form, is governed by actions to maximize a utility function, $U(E(x_T), Var(x_T))$ that satisfies

$$\begin{aligned} \frac{\partial U}{\partial E(x_T)} &> 0 && (1.4) \\ \frac{\partial U}{\partial Var(x_T)} &< 0 && (1.5) \end{aligned}$$

A general model for dynamic portfolio selection, $P(U)$, can be now formulated as

$$\begin{aligned} P(U) \quad &: \quad \max \quad U(E(x_T), Var(x_T)) && (1.6) \\ s.t. \quad x_{t+1} &= s_t x_t + \sum_{i=1}^{n} \left(e_t^i - s_t\right) u_t^i \quad t = 0, 1, 2, ..., T-1 && (1.7) \end{aligned}$$

Both problem formulations $E(w)$ and $(P(U))$ are difficult to be solved directly as variance minimization is a notorious nonseparable and nonconvex optimization problem [15, 16]. We hereby propose a solution framework using embedding, separation, and convexification schemes. This solution framework leads to the development of a solution algorithm of parametric iterative dynamic programming. Transformation techniques are applied to embed the original problem into a parametric auxiliary problem which is tractable and can be solved analytically using dynamic programming. By investigating the relationship between the solution of the original problem and the auxiliary problem, we can indentify the optimal condition with which a solution of the auxiliary problem attains the optimum solution of the original problem.

1.2 Organization Outline

Before we move to the development of a novel solution framework for dynamic portfolio selection with a mean variance formulation, we will briefly discuss in Chapter 2 the concept and methodology of the classical mean-variance portfolio theory. The advantages of Markowitz approach in risk management will be highlighted. Feedback control in stochastic environment will be also reviewed in Chapter 2. Dynamic programming will be studied in generating optimal feedback control policy. In chapter 3, we will perform a survey of the current literature concerning multiple-period portfolio analysis. Limitations on existing literature will be stated clearly. In Chapter 4, we present the theory and solution methodology for problems $E(w)$ and $(P(U))$. Problem $E(w)$ can be viewed as a special form of $P(U)$ when U is linear with respect to $E(x_t)$ and $Var(x_T)$. Verification with the results in the single period mean-variance formulation is provided. Numerical examples are calculated to illustrate the solution methodology. In Chapter 5, we extend the results in Chapter 4 to investigate situations where no riskless asset exists in the portfolio selection. Comparisons and numerical examples are also given. Conclusions and recommendations for further studies are presented in chapter 6.

Chapter 2

Literature Review

The material presented in this chapter is an interdiscipline study of the application of dynamic optimization techniques in investment analysis. This chapter is addressed to readers who are not familiar to both areas. The theory and the computational procedures of mean-variance modelling will be first discussed. The advantage of mean-variance approach will be highlighted. We will then briefly review the stochastic optimal control techniques, especially, the dynamic programming that is the most powerful approach in generating a closed-loop control policy.

2.1 Modern Portfolio Theory

It has been entered a critical and innovative phase of investment since the publication of Harry M. Markowitz's paper entitled "Portfolio Selection" [17]. The portfolio theory introduces quantitative and scientific approaches to risk management. The original work by Markowitz and its extensions by himself and other researchers[1, 9, 11, 18, 19, 20, 21, 22] are now regarded as the core of the investment analysis and are generally accepted as Modern Portfolio Theory. The success of the theory can be proven by the selection of Harry M. Markowitz for the 1990 Nobel Prize in economics. The modern portfolio theory has been gained great attentions not only from the investment professionals, economists and financial analysts, but also from the mathematicians, statisticians, operations researchers and financial engineers.

Investment can be simply considered as a trade-off between certain current wealth and uncertain future rewards. The investor gets rid of the current wealth, invests in some economic activities, and hopes to receive an attractive reward. Here the current wealth given-up is known while the future return is generally uncertain. The measurements of reward is usually by means of rate of return r, which is defined as

$$r = \frac{x_1 - x_0}{x_0} \tag{2.1}$$

where x_1 is the future reward at the end of an investment period and x_0 is the initial wealth. The rate of return r is a random variable as x_1 is generally random in nature. Because randomness plays such an important role, investors have to take the risk into consideration when making decisions for investment. Risk can be generally viewed as the chance of injury, damage or losses[9] and is somehow a subjective and psychological matter. Markowtiz introduced mean and variance of return rate as the quantitative measurements for return and risk, respectively. Based on these quantitative measurements, we are able to develop a systematic approach managing both return and risk while constructing our investment strategies.

2.1.1 Mean-Variance Model

Instead of considering individual securities separately, which was the so-called security analysis, Markowitz's portfolio analysis considers various securities in one basket at the same time. The portfolio analysis is trying to select the optimal allocation of wealth among all the individual securities under consideration. We hereby consider the return and risk of a portfolio as a whole rather than individual mean and the variance of each security. Some major assumptions on the attitude of investors and market operations are stated below.

Assumption 1 ***(Non-satiation)*** *: A rational investor will prefer a portfolio of higher return to that of lower return, provided that the all portfolios under consideration are bearing the same level of risk.*

Assumption 2 ***(Risk Aversion)*** *: A rational investor will prefer a portfolio of lower risk level to that of higher risk level, provided that the all portfolios under consideration are having the same return.*

Assumption 3 ***(Price Taker)*** *: Investors are considered as price-takers. They are unable to affect the price or the rate of return associated with any security no matter how much they invested.*

Assumption 4 ***(Infinite Divisibility)****: Securities can be divided in any unit, and therefore investor is possible to invest any desired amount in all securities.*

Assumption 5 ***(Symmetric Information)*** *: The market is assumed to be efficient and thus all investors are having the same information.*

Assumption 6 ***(Friction-less Transactions)*** *: The transactions in the markets are assumed to be cost-less and there is no commissions or taxes.*

Assumption 7 ***(No Arbitrage)*** *: The expected risk perimum for any risky asset over the risk-free asset should be positive. i.e.* $E(e - S) > 0$ *where* e *is the return for risky asset while* S *is the risk-less rate of return.*

The procedure of a mean-variance analysis can be divided into the following steps. We will further investigate these steps in detail.

Step 1 *Setting-up the relationship between the portfolio and its component securities.*

Step 2 *Identifying the efficient frontier of portfolios.*

Step 3 *Selecting the best compromised portfolio.*

2.1.2 Setting-up the relationship between the portfolio and its component securities.

Suppose there are n securities having rates of return $r' = [r_1, r_2, r_3, ..., r_n]$ following normal distribution with mean $E(r') = [E(r_1), E(r_2), E(r_3), ..., E(r_n)]$ and variance-covariance matrix $Cov(r) = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{bmatrix}$.

If we invest u_i in security i and define $u' = [u_1, u_2, u_3, ..., u_n]$, then we can setup the following relationship

$$\begin{aligned} E(r_p) &= u'E(r) && (2.2) \\ &= \sum_{i=1}^{n} u_i E(r_i) && (2.3) \end{aligned}$$

where $E(r_p)$ is the expected return of the portfolio . On the other hand, the risk of the portfolio is measured by the variance of the portfolio which is related to the component's variance and covariance

$$\begin{aligned} Var(r_p) &= u'Cov(r)u \\ &= \sum_{i=1}^{n} u_i^2 \sigma_{ii} + 2\sum_{i=1}^{n}\sum_{i \neq j}^{n} u_i u_j \sigma_{ij} && (2.4) \end{aligned}$$

where $Var(r_p)$ is the variance of the portfolio return while σ_{ii} is the variance of the return of security i and σ_{ij} is the covariance between the returns of securities

i and j. The following relationship exists between u_i, $i = 1, 2, ..., n$ and x_0.

$$\sum_{i=1}^{n} u_i = x_0 \tag{2.5}$$

2.1.3 Identifying the efficient frontier

Based on the assumptions 1 and 2, our objective is to allocate the wealth such that the portfolio offers the maximum expected return for given risk level or bears the minimal risk for given expected rate of return. We can formulate the portfolio selection problem using an expected return maximization model denoted by $P\,(\text{mean} : \nu)$,

$$P\,(\text{mean} : \nu) \quad : \quad \max_{u} \; u'E\,(r) \tag{2.6}$$

$$s.t. \; u'Cov\,(r)\,u \quad \leq \quad \nu \tag{2.7}$$

$$\sum_{i=1}^{n} u_i \quad = \quad x_0 \tag{2.8}$$

where $\nu \geq 0$ is a parameter specified by a particular investor.

Alternatively, we can formulate the portolio selelction problem using a variance minimization model, $P\,(\text{variance} : \varepsilon)$,

$$P\,(\text{variance} : \varepsilon) \quad : \quad \min_{u} \; u'Cov\,(r)\,u \tag{2.9}$$

$$s.t. u'E\,(r) \quad \geq \quad \varepsilon \tag{2.10}$$

$$\sum_{i=1}^{n} u_i \quad = \quad x_0 \tag{2.11}$$

where $\varepsilon \geq 0$ is a parameter given by a particular investor.

For both of the above formulations, we can construct an equivalent model, $P\,(\text{mean, variance} : w)$, by introducing a weighting coefficient w,

$$P\,(\text{mean, variance} : w) \quad : \quad \max_{u} \; u'E\,(r) - wu'Cov\,(n)\,u \tag{2.12}$$

$$s.t. \; \sum_{i=1}^{n} u_i \; = \; x_0 \tag{2.13}$$

where w (> 0) is specified by the particular investor.

Definition 1 *A portfolio, u, is said to be feasible if u satisfies (2.5).*

Definition 2 *A portfolio, u^*, is said to be efficient if there does not exist other feasible u such that $u'Cov\,(n)\,u \leq u^{*\prime}Cov\,(n)\,u^*$ and $u'E\,(r) \geq u^{*\prime}E\,(r)$ with at least one strict inequality.*

Definition 3 *As the value of ν, ε or w varies, the corresponding u^* is obtained. The locus of $u^{*\prime}E\,(r)$ and $u^{*\prime}Cov\,(n)\,u^{*\prime}$ in space $\{E\,(r_p)\,, Var\,(r_p)\}$ is called efficient frontier.*

Graphically, we can represent the feasible set and efficient frontier in the mean-variance space as in Figure 2.1.

2.1.4 Selecting the best compromised portfolio

The selection of a best-compromised portfolio is to make the indifference curve tangent to the efficient frontier. An indifference curve is the locus of points in the

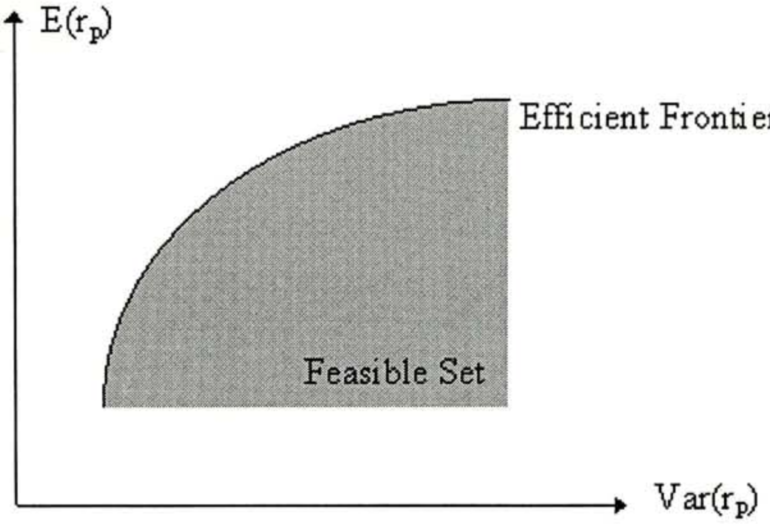

Figure 2.1: Feasible Set and Efficient Frontier in Mean-variance space

$(E(r_p), Var(r_p))$ space that have the same value of utility. In accordance to the assumptions 1 and 2 stated above, the indifference curve should have an upwards-sloping shape while the value of utility is increased when moving a north-west direction. Figure 2.2 depicts a map of indifferent curves where $U_3 > U_2 > U_1$.

The best-compromised portfolio is therefore the one lying on the efficient frontier while having the highest possible utility function value. One can find it out by pushing the indifference curves out of the feasible region. The point where the efficient frontier is tangent to the indifference curve is the best-compromised portfolio for the investor as shown in Figure 2.3.

The mean-variance formulation has several features that are very important in theoretical development and practical computations. First of all, mean-variance is a good approximation of expected utility maximization[2, 3, 4, 5, 10]. Secondly,

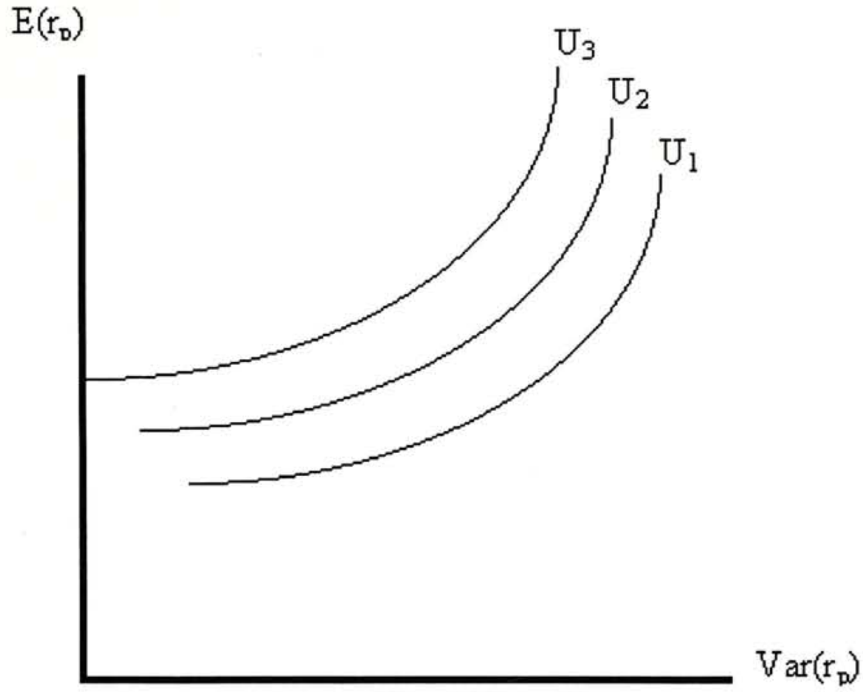

Figure 2.2: Map of Indifference Curves

since we assume all investors in the market are having the same information (assumption 5), the efficient frontier generated by an analyst will be the same for all investors. Therefore, we only need to compute one efficient frontier while investors are of different attitudes towards risk. For investors of different degree of risk aversion, we can simply match his indifference curve with the unique efficient frontier whereas the utility maximization approach has to re-work for different investors. Thirdly, as we can see, the dominant part in portfolio risk is related to the covariance terms while variance-associated terms are less important. If the number of risky securities is large, variance terms become negligible[1, 21]. This important finding suggests that unless the coefficient of correlation is positive one, diversification can always reduce risk.

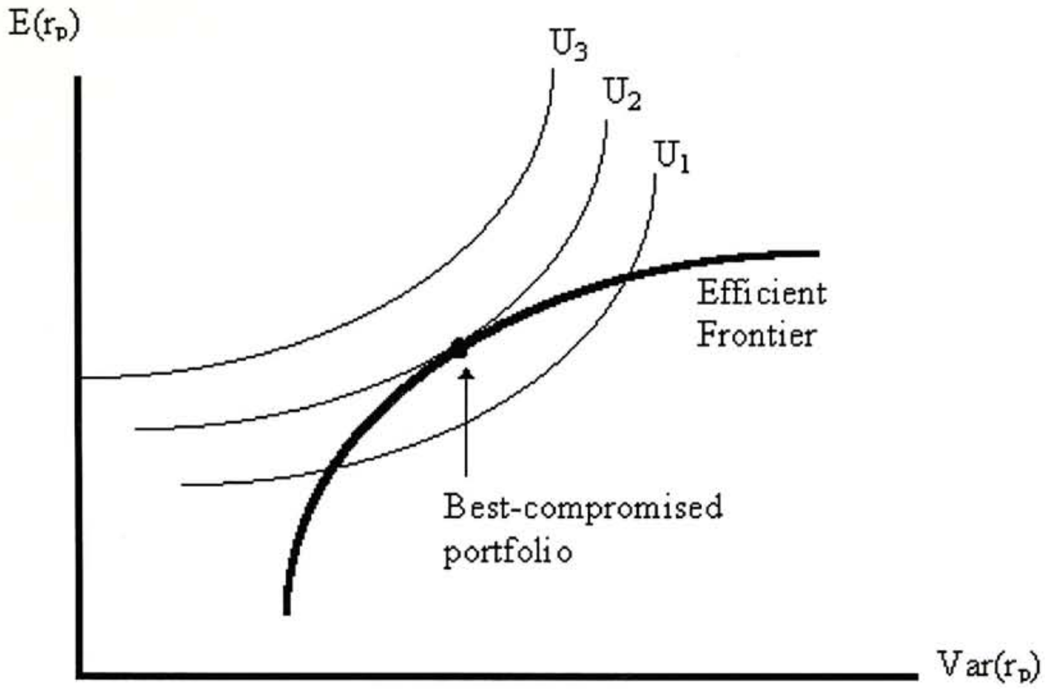

Figure 2.3: Best-compromised Portfolio

In parctical implementations, it is very difficult to generate a smooth efficient frontier when an analytical solution is impossible. A method called critical line method proposed by Markowitz [17] approximates the efficient frontier by employing quadratic programming repeatedly for a problem similar to P (mean,variance) with different values of w. The efficient frontier is then approximated by the line segements joining the optimal points obtained for different w in the mean-variance space. A simplified model is provided by Sharpe[21] in order to save computational efforts. Merton[23] finally developed the analytic form of the efficient frontier in 1972. Another direction in mean-variance portfolio optimization research is to make the existing model more realistic. The formulations have the similar objective functions but incorporate more realistic constraints such as a limit for short-selling, lower and upper bounds on investment, and transaction costs [18].

2.2 Stochastic Optimal Control

Optimal control is to find a control law that optimizes a given criterion for a given dynamic process. One example is the following deterministic continuous optimal control problem given by Åström[24]

$$\min_u J = \int_0^\infty \left[x_t^2 + u_t^2\right] dt \tag{2.14}$$

$$\frac{dx_t}{dt} = u_t \tag{2.15}$$

with initial condition

$$x_0 = 1 \tag{2.16}$$

Optimal control policy can be classified into two classes by the strategies to map the acquired information into the control law : open-loop control law and closed-loop (feedback) control law. Open-loop control law maps priori data to form the control strategy whereas feedback control maps the current state of the process to form the control strategy. From another point of view, we can classify the control problems by the nature of dynamic systems. If there is no stochastic disturbance in the systems equations, we can call it a deterministic problem, otherwise, it is called a stochastic optimal control problem.

The derived open-loop control and closed-loop control cannot be distinguished in deterministic problems. Consider the above problem again. The open-loop control law obtained is

$$u_t = -e^{-t} \tag{2.17}$$

whereas the closed-loop control is

$$u_t = -x_t \tag{2.18}$$

Notice that $x_t = e^{-t}$ under the control $u_t = -e^{-t}$. Thus the open-loop control and the closed-loop control are the same. Both control rules give the same value, $J = 1$. However, in cases where there are disturbances in the process, only the closed-loop control can generate an optimal control strategy. A derviation from an nominal trajectory can be adaptively corrected by feedback control while deviations will be cumulated in an open-loop control fashion.

This limitation in open-loop control policy has been recognized at the very beginning of the development in control theory[24]. The solution for closed-loop optimal control has been developed, while heavily depending on the development of the theory of dynamic programming [24][25]. Indeed, dynamic programming is regarded as the universal solution scheme to generate the feedback control policy for stochastic systems.

2.2.1 Dynamic Programming

Dynamic programming was pioneered by Richard Bellman in 1950's. Since then dynamic programming has been widely applied in many fields such as decision science, control engineering, economics and finance, optimization and so on. Unlike other mathematical programming techniques, dynamic programming is an approach instead. We solve complex problem by decomposing it into a chain of sub-problems that are easier to be solved. Then we composite the results to yield the solution for the original complicated problem. Dynamic programming has now been ranked as one of the most powerful optimization techniques as the approach can be applied to problem of different structure : discrete and continuous, linear and nonlinear, deterministic and stochastic. Another reason for wide applications of dynamic programming is due to the rapid development of digital computer. The recursive structure of dynamic programming can be easily converted into programming codes. The most important concept for dynamic programming is Bellman's principle of optimality[26].

Principle of Optimality An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decisions.

In other words, instead of searching the optimal soluition from all feasible solutions, we only need to consider those solutions that satisfy the Principle of Optimality. Consider a discrete-time optimal control problem,

$$\min \quad J = \Phi_T(x_T) + \sum_{t=0}^{T-1} \Phi_t(x_t, u_t) \tag{2.19}$$

$$s.t. \quad x_{t+1} = f_t(x_t, u_t) \quad t = 0, 1, 2..., T-1 \tag{2.20}$$

$$x_0 \text{ is given}$$

We embed the original problem into a family of discrete-time optimization problems starting at period i with state x_i. Hence, we can define a cost-to-go function from period i with state x_i to the last period T as $J_i(x_i)$,

$$\begin{aligned} \min \quad J_i(x_i) &= \Phi_T(x_T) + \sum_{t=i}^{T-1} \Phi_i(x_i, u_i) &\quad (2.21) \\ s.t. \quad x_{t+1} &= f_t(x_t, u_t) \quad t = i, i+1, ..., T-1 &\quad (2.22) \end{aligned}$$

$$x_i \text{ is given}$$

Define optimal cost-to-go from x_i as

$$J_i^*(x_i) = \min_u \quad J_i(x_i) \tag{2.23}$$

Note that

$$\begin{aligned} J_i^*(x_i) &= \min\left(\Phi_T(x_T) + \sum_{t=i}^{T-1} \Phi_i(x_i, u_i)\right) \\ &= \min\left(\Phi_i(x_i, u_i) + \Phi_T(x_T) + \sum_{t=i+1}^{T-1} \Phi_{i+1}(x_{i+1}, u_{i+1})\right) \\ &= \min\left(\Phi_i(x_i, u_i) + J_{i+1}^*(x_{i+1})\right) \end{aligned} \tag{2.24}$$

The above recursive relationship is the direct application of Principle of Optimality. The optimal cost-to-go at period t can be divided into two components, cost from x_t to x_{t+1} and the optimal cost-to-go from x_{t+1} to x_T. Implementation of dynamic programming requires a boundary condition. Stochastic dynamic programming problems are solved in a backward recursion. The general boundary condition is

$$J_T^*(x_T) = \Phi_T(x_T) \tag{2.25}$$

2.2.2 Dynamic Programming Decomposition

The additive form in objective function is just one of the common forms. Without loss of generality, we can consider the following dynamic optimization problem such that

$$\begin{aligned} \min_{u_1,u_2,\ldots,u_{T-1}} \quad J &= g_1\left[\Phi_1(x_1,u_1), \Phi_2(x_2,u_2), \ldots, \Phi_T(x_T)\right] & (2.26) \\ s.t. \quad x_{t+1} &= f_t(x_t,u_t) \quad t = 0,1,2\ldots,T-1 & (2.27) \\ & x_0 \text{ is given} \end{aligned}$$

Dynamic programming approach decomposes the objective function into a series of chained sub-functions such that each sub-problem is just a single-period optimization problem. To achieve this goal, separable property for the cost-to-go function g_1 is requried.

Definition 4 *(Separability[27]) For any control policy, the cost-to-go at every state must be expressible as a function of the immediate cost and the cost-to-go at a succeeding state. In other words, The function g_1 is separable if it can be expressed by*

$$\begin{aligned} & g_1\left[\Phi_1(x_1,u_1), \Phi_2(x_2,u_2), \ldots, \Phi_T(x_T)\right] & (2.28) \\ = \; & g_1\left[\Phi_1(x_1,u_1), g_2\left[\Phi_2(x_2,u_2), \ldots, \Phi_T(x_T)\right]\right] & (2.29) \end{aligned}$$

Dynamic programming requires that all the cost-to-go functions be separable so that recursive operations can be carried out. In stochastic environments, it is generally required that the process satsifies Markovian Property. i.e. f_t is a transformation such that x_{t+1} is only relied on x_t and independent of state before t.

As we have discussed, a little change in state during the process can cause open-loop optimal law to fail. Feedback control is thus essential for a stochastic system as it generates decision rule based on the current state information and it is capable of on-line adapting the changes. Dynamic programming is thus one of the most powerful tools for generating closed-loop control policy. However, the separablity condition must be satisfied for decomposition of a recursive relation. Recent research results, Sniedovich [16] and Li[28, 29], extend dynamic programming to tackle non-separable problems such as variance minimization problems.

Chapter 3

Multiple Period Portfolio Analysis

Multiple period (or multi-period for short) portfolio theory can be classified into several categories depending on the objective function. The individual investors invest in capital markets, adjust their investment strategies in the intermediate periods and withdraw the payoff for intermediate period consumptions. Their investment behaviour could be determined by maximizing an utility of multi-period consumption[7]. While, on the other side, most of institutional investors such as pensions funds, mutual funds, and the trust department in banks make investment decision for a specific purpose at the end of a planning horizon. Intermediate consumptions are not considered in these cases. This kind of behaviour can be described by the models that maximize a utility of final wealth[6]. These two kinds of models are the traditional formulation in the early studies of multi-period investment analysis and were summarized by Elton and Gruber[8]. Thirdly, a model proposed by Hankansson[14] adopts an objective function that maximizes the expected average compound return. The fourth kind is to set an investment strategy such that targeted capital is reached as early as possible. The problem is formulated in Heath *el at* [30] to minimize the time to reach that target. Last but not least, there are some investors who concern about the possibility for the investment process to keep above a certain survival wealth level. Such kind of behaviour is formulated in a goal seeking investment model by Cogger *el at* [31].

Except those pointed out in the particular discussion, the notations in Table 3.1 are commonly used in the following chapters.

3.1 Maximization of Multi-period Consumptions

One of the proper multi-period or long run investment formulations is to assume that an investor is trying to maximize the utility associated with his or her consumption at various periods, $U(c_0, c_1, c_2, .., c_{T-1})$, by making multi-period decisions of investments $u_t = [u_t^1, u_t^2, ..., u_t^n]$ and consumptions c_t[7][8]. The objective function can be expressed as the expected sum of discounted utility of consumptions. The discount rate is assumed to be a constant ρ. It is also assumed that the last period consumption will be the amount left at period T, i.e. $c_T = x_T$.

$U(\cdot)$	Utility function
c_t	Consumption decision for time period t
u_t	Investment decision vector for time period t, i.e. $u_t = [u_t^1, u_t^2, ..., u_t^n]$
u_t^i	Investment decision of security i for time period t
e_t	Return vector of securities for period t, i.e. $e_t = [e_t^1, e_t^2, ..., e_t^n]$
e_t^i	Return of security i for period t
s_t	Return for risk-free asset in period t
n	Total number of periods under consideration
T	Total number of period under consideration
x_t	Wealth at the beginning of period t
P_t	Return difference between risky and risk-free assets at period t, i.e. $P_t^i = e_t^i - s_t$

Table 3.1: Notations

Consider a general problem that an investor can allocate his wealth into a set of n risky assets and/or a risk-less asset offering a certain return rate of s_t at time t. The rates of return of the risky assets at any time t, $e_t = (e_t^1, e_t^2, e_t^3, .., e_t^n)'$, are following a multi-variate distribution with mean, $E(e_t) = [E(e_t^1), E(e_t^2), ..., E(e_t^n)]$, and variance-covariance matrix,

$$Cov\left(e_t\right) = \begin{bmatrix} \sigma_{t,11} & \sigma_{t,12} & \cdots & \sigma_{t,1n} \\ \sigma_{t,21} & \sigma_{t,22} & \cdots & \sigma_{t,2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{t,n1} & \sigma_{t,n2} & \cdots & \sigma_{t,nn} \end{bmatrix}$$

The wealth dynamic between any two successive periods can be described by the following equation,

$$\begin{aligned} x_{t+1} &= \left(x_t - c_t - \sum_{i=1}^{n} u_t^i\right) s_t + \sum_{i=1}^{n} u_t^i e_t^i \qquad (3.1) \\ &= \left(x_t - c_t\right) s_t + P_t' u_t \qquad (3.2) \end{aligned}$$

Hence, we can have the following expression,

$$c_t = x_t - \frac{x_{t+1} - P_t' u_t}{s_t} \qquad (3.3)$$

where $P_t = \left(e_t^1 - s_t, e_t^2 - s_t, e_t^3 - s_t, .., e_t^n - s_t\right)' \; \forall t$. We can then have the following mathematical model in maximizing multi-period consumptions,

$$J_0 = \max_{(c_1,u_1),(c_2,u_2),\ldots,(c_{T-1},u_{T-1})} E\left(\sum_{t=0}^{T} (1+\rho)^{-(T-t)} U\left(c_t\right)\right) \qquad (3.4)$$

$$s.t. \quad c_t = x_t - \frac{x_{t+1} - P_t' u_t}{s_t} \qquad t = 0, 1, ..., T-1 \qquad (3.5)$$

$$c_T = x_T \qquad (3.6)$$

x_0 given.

Note that the expectation $E\left(\cdot\right)$ operates only on random variables in the future periods since current consumption is known once our decision is made.

To determine the consumption and investment decisions, we have to solve the problem backward from period $T-1$. Note that

$$c_T = x_T \tag{3.7}$$

$$c_T = (x_{T-1} - c_{T-1})\, s_{T-1} + P'_{T-1} u_{T-1} \tag{3.8}$$

and $J_T(x_T) = U(c_T)$.

We first consider the cost-to-go function for period $T-1$.

$$J_{T-1}(x_{T-1})$$

$$= \max_{c_{T-1}, u_{T-1}} U(c_{T-1}) + (1+\rho)^{-1} E(U(c_T)) \tag{3.9}$$

$$= \max_{c_{T-1}, u_{T-1}} U(c_{T-1}) + \left(\frac{1}{1+\rho}\right) E(J_T(x_T)) \tag{3.10}$$

$$= \max_{c_{T-1}, u_{T-1}} U(c_{T-1}) + \left(\frac{1}{1+\rho}\right) E\left(J\left((x_{T-1} - c_{T-1})\, s_{T-1} + P'_{T-1} u_{T-1}\right)\right) \tag{3.11}$$

Taking partial differentiation with respect to c_{T-1} and u_{T-1} respectively, we have the optimal conditions by setting $\frac{\partial J_{T-1}}{\partial c_{T-1}} = 0$, i.e.

$$U'(c_{T-1}) - \left(\frac{s_{T-1}}{1+\rho}\right) E(J'(x_T)) = 0 \tag{3.12}$$

and $\frac{\partial J_{T-1}}{\partial u_{T-1}} = 0$, i.e.

$$E(J'_T(x_T))\, P_{T-1} = 0 \tag{3.13}$$

When $t = T-2$, the cost-to-go function at time $T-2$ is

$$J_{T-2}(x_{T-1})$$

$$= \max_{c_{T-2},u_{T-2}} U(c_{T-2}) + (1+\rho)^{-1} E(U(c_{T-1})) \tag{3.14}$$

$$= \max_{c_{T-2},u_{T-2}} U(c_{T-2}) + \left(\frac{1}{1+\rho}\right) E(J_{T-1}(x_{T-1})) \tag{3.15}$$

$$= \max_{c_{T-2},u_{T-2}} U(c_{T-2}) + \left(\frac{1}{1+\rho}\right) E\left(J_{T-1}\left((x_{T-2} - c_{T-2}) s_{T-2} + P'_{T-2} u_{T-2}\right)\right) \tag{3.16}$$

The optimality conditions of $\frac{\partial J_{T-2}}{\partial c_{T-2}} = 0$ and $\frac{\partial J_{T-2}}{\partial u_{T-2}} = 0$ are

$$U'(c_{T-2}) - \left(\frac{s_{T-2}}{1+\rho}\right) E\left(J'_{T-1}(x_{T-1})\right) = 0 \tag{3.17}$$

and

$$E\left(J'_{T-1}(x_{T-1})\right) P_{T-2} = 0 \tag{3.18}$$

respectively. In general, we can derive the recursive equations as the following,

$$U'(c_t) - \left(\frac{s_t}{1+\rho}\right) E\left(J'_{t+1}(x_{t+1})\right) = 0 \tag{3.19}$$

and

$$E\left(J'_{t+1}(x_{t+1})\right) P_t = 0 \tag{3.20}$$

where $t = 0, 1, ..., T-1$ and the expectation $E(\cdot)$ operates only on the random variable of the future periods.

However, if no additional assumption is made about the form of $U(c_t)$, this solution form is of little use[8]. It is usually assumed that the utility function of consumptions, $U(c_0, c_1, c_2, ..., c_{T-1})$, are separable with respect to $c_0, c_1, c_2, ..., c_{T-1}$. Under this assumption, the utility of consumption at any time t is unrelated to any past or future periods. The problem satisfies the separablity condition and can be solved by stochastic dynamic programming.

3.2 Maximization of Utility of Terminal Wealth

Many trust funds are managed in order to maximize a utility in terms of the final wealth only. The funds set-up by investors generally have a known time-horizon and specific purposes such as purchasing a house, taking a long vacation after retirement are sought from this financial funding. Essentially, as pointed out by Francis[9], it is economically equivalent between maximizing the utility of terminal wealth and maximizing the expected sum of consumptions. Since, source for intermediate consumptions can be a loan from bank. If the portfolio grows, the investor can borrow more for consumption by pledging the portfolio as a collateral. A mathematical model in maximizing a utility of terminal wealth can be posted as follows

$$\max E\left(U\left(x_T\right)\right) \tag{3.21}$$

$$s.t. \quad x_{t+1} = s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 \tag{3.22}$$

where x_t is the value of the investor's portfolio at the beginning of t, x_T is terminal welath and $U\left(\cdot\right)$ is the utility function.

Let $J_t\left(x_t\right)$ be the derived expected utility from period t to the final stage horizon T for given x_t dollars. We can build the following recursive equations for

$J_t(x_t)$.

$$J_t(x_t) = \max_{u_t} E\left(J_{t+1}(x_{t+1})\right) \tag{3.23}$$

$$s.t.\ x_{t+1} = s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 \tag{3.24}$$

with the following boundary condition

$$J_T(x_T) = U(x_T) \tag{3.25}$$

There are some typical forms of utility functions that are widely used by researchers. The most popular utility function is the quadratic form as the following,

$$U(x_T) = x_T - w x_T^2 \quad \text{for some } w > 0 \tag{3.26}$$

Solving the formulation (3.23)-(3.24) recursively, the optimal investment law can be obtained.

$$u_t = \frac{1}{2} E\left(P_t P_t\right)^{-1} \left(w_t - 2 s_t x_t\right) E\left(P_t\right) \tag{3.27}$$
$$t = 1, 2..., T-1$$

where

$$w_{t+1} = w_t s_t \left(1 - E(P_t) E\left(P_t P_t\right)^{-1} E(P_t)\right) \tag{3.28}$$
$$t = 1, 2..., T-1$$

with the following bounday condition,

$$w_T = w \tag{3.29}$$

Mossin[6] investigated this problem deeply. It could be seen that the coefficient w decreases as time increases. This effect was called "time effect". In other words, investment in risky assets will be larger when the time horizon is coming closer. It could be understood that the investors need to predict less not need to predict less when getting closer to the time horizon. They therefore are willing to shift their wealth from the riskfree asset to risky asset so as to earn a higher return.

There are suitations where the optimal investment decisions are completely myopic. A neccessary and sufficient condition was derived by Mossin in [6].

$$\frac{-U'(x_T)}{U''(x_T)} = kx_T \tag{3.30}$$

Typical examples of utility functions that satisfy this condition are $U(x_T) = \ln(x_T)$, $U(x_T) = E\left((x_T)^{1-w}\right)$ where $0 < w < 1$. In these cases, the optimum proportion of investments in different assets is independent of the wealth at any current period[8, 12, 13]. Consequently, we use u_t^i to denote the percentage of x_t that is invested in the i-th risky asset at time t. The actual investment will be $x_t u_t^i$ for security i at time t[8]. Thus, the wealth dynamic can be rewritten as the following for any time t

$$x_{t+1} = (s_t + P_t' u_t)\, x_t, \qquad t = 0, 1, ..., T-1 \tag{3.31}$$

Now consider a situation where an investor seeks optimal investment strategies u_t, for all $t = 0, 1, ..., T-1$, such that a log-form utility function is maximized

$$\max_{u_1, u_2, ..., u_{T-1}} U(x_T) = \ln(x_T) \tag{3.32}$$

We can solve the problem by dynamic programming starting from $T-1$. At stage T-1, we have

$$\max_{u_{T-1}} J_{T-1}(x_{T-1}) = E(\ln(x_T)) \tag{3.33}$$

$$s.t.\ x_T = s_{T-1}x_{T-1} + P'_{T-1}u_{T-1}x_{T-1} \tag{3.34}$$

Then we are going to maximize the derviced utility function, i.e.

$$\begin{aligned}\max_{u_{T-1}} J_{T-1}(x_{T-1}) &= E\left(\left[\ln\left(s_{T-1} + P'_{T-1}u_{T-1}\right)x_{T-1}\right]\right) \\ &= E\left(\ln\left(s_{T-1} + P'_{T-1}u_{T-1}\right)\right) + \ln(x_{T-1}) \end{aligned} \tag{3.35}$$

In general, at time t, the investor is to going to maximize the following dervied utility function,

$$\max_{u_t,u_{t+1},\ldots,u_{T-1}} J_t(x_t) = \ln(x_t) + E\left(\sum_{i=t+1}^{T} J_i(1)\right) \tag{3.36}$$

where $J_i(1) = E(\ln(P'_i u_i + s_i)) = E(U(P'_i u_i + s_i))$, which is the expected value of the utility obtained by investing a dollar from the beginning of period i to the beginning of period $i+1$.

Another form of the utility function, that has been widely used in the literature to derive optimal myopic policies, is the power-form utility function. Similar to the situation of log-form utility function, the wealth at time $t+1$ is a multiplier of the wealth at time t[8]. The investor is seeking for optimal control policy such that of the following stochastic maximization problem is solved,

$$\max_{u_1,u_2,\ldots,u_{T-1}} \quad U(x_T) = E\left((x_T)^{1-w}\right) \qquad 0 < w < 1 \tag{3.37}$$

$$s.t.\ x_{t+1} = \left(P_t' u_t + s_t\right) x_t \qquad t = 0, 1, ..., T-1 \tag{3.38}$$

Define the cost-to-go function at time $T-1$ as

$$\max_{u_{T-1}} \quad J_{T-1}\left(x_{T-1}\right) = E\left(x_T\right)^{1-w} \tag{3.39}$$

$$s.t.\ x_T = \left(P_{T-1}' u_{T-1} + s_{T-1}\right) x_{T-1} \tag{3.40}$$

i.e.

$$\begin{aligned} \max_{u_{T-1}} \quad J_{T-1}\left(x_{T-1}\right) &= E\left(\left(P_{T-1}' u_{T-1} + s_{T-1}\right) x_{T-1}\right)^{1-w} \\ &= x_{T-1}^{1-w} \cdot E\left(P_{T-1}' u_{T-1} + s_{T-1}\right)^{1-w} \end{aligned} \tag{3.41}$$

Generally, at any time t, the cost-to-go function is of the following form,

$$J_t\left(x_t\right) = \max_{u_t, u_{t+1}, \ldots, u_{T-1}} x_t^{1-w} \left(\prod_{i=t+1}^{T} J_i\left(1\right)\right) \tag{3.42}$$

where $J_i\left(1\right) = E\left(P_i' u_i + s_i\right)^{1-w} = E\left(U\left(P_i' u_i + s_i\right)\right)$, which is the expected utility obtained from investing a dollar at the beginning of period i to the beginning of period $i+1$.

3.3 Maximization of Expected Average Compounded Return

Hakansson[14] considered a multi-period portfolio selection model by maximizing the expected average compound return over T periods. In his model, he assumed that the amount of investments are independent of the initial wealth at the beginning of that period. Hence, instead of considering the amount of investments, we now are considering the porportions. Denote the proportion of wealth invested in asset i at period t be u_t^i. The actual amount invested will be $u_t^i x_t$ for wealth x_t at the beginning of the period. The wealth dynamic can be simplified in the following

$$x_{t+1} = \left(P_t' u_t + s_t\right) x_t \qquad t = 0, 1, ..., T-1 \tag{3.43}$$

Since P_t' and x_t are independent, we have

$$E\left(x_{t+1}\right) = E\left(P_t' u_t + s_t\right) E\left(x_t\right) \tag{3.44}$$

The following is straightforward,

$$E\left(x_{t+1}\right) = x_0 E \prod_{k=1}^{k=t} \left(P_k' u_k + s_k\right) \qquad t = 0, 1, ..., T-1 \tag{3.45}$$

When the distribution of returns are independent over time, we can have the following

$$E\left(x_{t+1}\right) = x_0 \prod_{k=1}^{k=t} E\left(P_k' u_k + s_k\right) \qquad t = 0, 1, ..., T-1 \tag{3.46}$$

In other words, the investment objective of maximizing expected average compounded return, i.e.

$$\max_{u_t} \quad E\left(\prod_{t=1}^{T} x_t\right)^{\frac{1}{T}} \tag{3.47}$$

can be rewritten as

$$\begin{aligned} & \max \quad x_0 \prod_{t=1}^{T} E\left(x_t^{\frac{1}{T}}\right) \\ = & \max \quad x_0 E\left(x_1^{\frac{1}{T}}\right) E\left(x_2^{\frac{1}{T}}\right) ... E\left(x_T^{\frac{1}{T}}\right) \end{aligned}$$

It is obvious that the objective function is separable and dynamic programming techniques can directly be applied. Recurssive function can be easily obtained such that

$$J_t^*(x_t) = \max_{u_t} \quad x_0 E\left(x_t^{\frac{1}{T}}\right) J_{t+1}^*(x_{t+1})$$

Since the deduced utility function at each period is strictly concave, it implies the risk averison and thus diversifications. Hakansson claimed that the risk is not necessary to be considered explicitly as a trade-off to expected return because diversification is a must for anyone who is interested in maximizing expected average compounded return alone a time horizon with more than one period.

3.4 Minimization of Time to Reach Target

A portfolio selection problem is discussed by Heath *et al* [30] where minimum time to reach a predetermined target by an investor is investigated. Techniques using diffusion process and Ito's calculus are applied in construction of investment strategies. The problem is to manage a portfolio containing a stock and a bond so as to minimize the expected time to reach a given total worth.

The bond price b_t is supposed to follow the following process,

$$db_t = r_b b_t dt. \tag{3.48}$$

where r_b is the rate of return for the bond and is assumed to be positive. The stock price e_t is assumed to satisfy the following process

$$de_t = r_s e_t dt + \sigma_e e_t dZ_t \tag{3.49}$$

where r_s is the rate of return for the stock and Z_t is a standard Brownian motion with variance σ_e. The total wealth at time t is thus determinated by,

$$dx_t = x_t \left(r_s u_s(t) + r_b u_b(t)\right) dt + \sigma_e u_s(t)\, dZ_t \tag{3.50}$$

The following optimal control policy is found using the theorems derived by Heath *et al* [30]

$$u_s = \begin{cases} \frac{r_s - r_b}{\sigma_e^2} & \text{if this is less than 1} \\ 1 & \text{otherwise} \end{cases} \tag{3.51}$$

$$u_b = 1 - u_s \tag{3.52}$$

where u_s and u_b are investment proportions in stock and bond, respectively. As we can observe from the optimal policy, the investor is trying to put as much as possible into stock market when $\frac{r_s - r_b}{\sigma_e^2} \geq 1$. In other words, if the "excess return" of $(r_s - r_b)$ is more than the risk offered by the stock, σ_e^2, investor will invest all of his wealth in the risky stock. On the other hand, if the "excess return" cannot compensate the risk by the stock, investor puts the maximum he is willing to bear, i.e. $\frac{r_s - r_b}{\sigma_e^2}$ while investing the rest into the bond market. As a result, the investor is investing at the maximum of his risk-bearing so as to minimize the time to reach the target. He will take the maximum risk without any safeguard to reach the objective whenever the "excess return" is high.

3.5 Goal-Seeking Investment Model

There is, in the literaute, another kind of human behaviour of goal seeking in investment. The goal is to maximize the probability of achieving a targeted rate of return by the end of the planning horizon. The concern from this kind of behavior is related not only to the wealth, but also to the time to reach the target.

The value of portfolio is described by the following stochastic process.

$$d(\ln x_t) = \mu dt + \sigma Z_t \sqrt{dt} \tag{3.53}$$

Equation (3.53) implies that $\ln x_t$ is a Wiener process with mean μt and variance $\sigma^2 t$. Thus x_t is log-normally distributed.

Assuming that, an investor has a wealth of x_0 at the beginning of planning horizon. The future value of principal grows at a rate of r. The investor sets a target present value, $\mathcal{X}$, to be achieved within a planning horizon of T. The value $\mathcal{X}$ is assumed to be greater than x_0. Then the overall goal is said to be achieved if and only if there is a $t \leq T$ such that $x_t \geq \mathcal{X} e^{rt}$. Or equivalently,

$$\ln x_t - rt \geq \ln \mathcal{X} \tag{3.54}$$

If $\ln x_t$ is a Wiener process, then $\ln x_t - rt$ will also be a Wiener process with mean $(\mu - r)\, t$ and variance $\sigma^2 t$.

It has been found that the probability of achieving the investment goal within the planning horizon T can be obtained from the Wald distribution F,

$$F(T, M, \sigma, R) = \Phi\left(\frac{MT - R}{\sigma\sqrt{T}}\right) + \left[\exp\left(\frac{2MR}{\sigma^2}\right)\right]\Phi\left(\frac{-MT - R}{\sigma\sqrt{T}}\right) \tag{3.55}$$

where

$$M = \mu - r \tag{3.56}$$

$$R = \ln\left(\frac{\mathcal{X}}{x_0}\right) \tag{3.57}$$

and Φ is the cumulative standard normal distribution.

Cogger *et al* [31] provide a sensitivity analysis for the four parameters of planning T, mean appreciation rate M, variance σ^2 and the targeted appreciation R. There are totally six pairs of trade-off analysis for these parameters. However, there is no model or technique used in Cogger *et al* 's paper for investment decision making. It seems that multi-attribute decision making techniques such as Analytical Hierarchy Process (AHP), or Surrogate Worth Trade-off method (SWT) and so on can be applied to help decision makers to figure out their strategies.

It is found that the results of goal seeking obtained may not maximize the wealth either in long term or in short time[31]. Unlike the situation discussed in the section of minimizing time to reach target, investors do not need to be so aggressive. They can take some safer investment policies (but lower return) to reach the pre-selected goal although the time taken will be longer (but within the time horizon). Therefore, in a short-term sense, we can then understand that the short-run wealth may not be necessarily maximized. While, on the other hand, there is a fixed goal to be acheived. The investors are not required to take any highly risky actions (but high return) to overshoot the pre-selected goal. In this sense, we may say that the long term wealth is not necessarily maximized as well.

Chapter 4

Multi-period Mean-Variance Analysis with a Riskless Asset

4.1 Motivation

From our survey, it is evident that there exist the following limitations in the existing literature of multi-period portfolio analysis.

1. The utility is usually assumed to be separable in periods. This does not always match the real world. For example, our consumption habbits are usually correlated over time. The consumption pattern in the current period will be affected by consumptions in the previous periods.

2. The resulted results are mainly myopic strategies. The decision is made without concerning about the information of future prediction. Although

stock prices are extremely difficult, if not impossible, to be forecasted in short run, there exists long-term cyclic and periodic patterns. Intelligent investors should take these factors into account.

3. The formulation requires an investor to assign the parameters in his/her utility. This is unrealistic and difficult for the investor to specify his/her risk aversion explicitly.

   In fact, most investors can only tell how much risk they can bear and/or how much return much get so as to survive. Therefore, multi-period formulations similar to $P\,(\text{mean} : \nu)$ and $P\,(\text{variance} : \varepsilon)$ sound more realistic.

4. For investors of different degrees of risk aversion, it needs to reformulate the problem and recalculate everything for each investor when adopting a utility maximization method.

One of the practical ways to construct utility is based on trade-off and comparison[19]. We feel puzzled why the mean-variance approach has received little attention in long-term multiple period analysis. So far, there is no published analytical results or an efficient numerical algorithm for deriving the efficient frontier for portfolio selection in a multi-period environment. In other words, the concept of the Markowitz's mean-variance approach has not yet been fully utilized in dynamic portfolio selection. One of the reasons may be due to the difficulties dealing with the variance term when adopting dynamic programming as a solution scheme. It is known that the variance-minimization is a non-separable problem in the sense of dynamic programming[15, 16]. The motivation of this research is to develop an efficient solution framework that can extend the existing literature to capture the spirit of risk management in dynamic portfolio selection.

Let us state clearly the problem of portfolio management in a dynamic setting. We are considering a capital market with n risky assets, with random rate of returns, and a risk-less asset offering a sure rate of return. An investor joins the market at time 0 with an initial wealth x_0 . The investor can allocate his wealth among the n risky assets and the risk-less asset. The current wealth can be reinvested at the beginning of each of $T-1$ consecutive time periods. The rates of return of the risky assets at any time t, $e_t = (e_t^1, e_t^2, e_t^3, .., e_t^n)'$ is following a multivariate distribution characterized by $E(e_t)$ and variance-covariance matrix,

$$Cov(e_t) = \begin{bmatrix} \sigma_{11,t} & \sigma_{12,t} & \cdots & \sigma_{1n,t} \\ \sigma_{21,t} & \sigma_{22,t} & \cdots & \sigma_{2n,t} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1,t} & \sigma_{n2,t} & \cdots & \sigma_{nn,t} \end{bmatrix}.$$

We define $P_t = (e_t^1 - s_t, e_t^2 - s_t, e_t^3 - s_t, .., e_t^n - s_t)'$ for $t = 0, 1, ..., T-1$. Since $E(e_t'e_t)$ is always positive defined, we can have $\begin{bmatrix} s_t^2 & s_t e_t' \\ s_t e_t & E(e_t e_t') \end{bmatrix}$ is positive definite for all time periods. Then, the following is true.

$$\begin{bmatrix} s_t^2 & s_t E(P_t') \\ s_t E(P_t) & E(P_t P_t') \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} s_t^2 & s_t E\left(P_t'\right) \\ s_t E\left(P_t\right) & E\left(P_t P_t'\right) \end{bmatrix} \begin{bmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} > 0 \quad (4.1)$$

Further we have the following,

$$E(P_t P_t') > 0, \forall t$$

and

$$1 - E(P_t')E^{-1}(P_t P_t')E(P_t) > 0, \forall t \quad (4.2)$$

In this chapter, we will first investigate utility that is in a linear form with respect to the mean and the variance. This leads to the derivation of an analytical expression of the mean-variance efficient frontier for multi-period portfolio selection. We will then further extend the result to cover a broader class of nonlinear utility.

4.2 Dynamic Mean-Variance Analysis Formulation

The objective of an investor is to maximize the expected terminal wealth $E\left(x_T\right)$, subject to that the variance of the terminal wealth, $Var\left(x_T\right)$, is not greater than a pre-selected value. Mathematically, one can formulate the problem as,

$$P(\nu): \quad \max\ E(x_T) \tag{4.3}$$

$$s.t.\ Var(x_T) \leq \nu \tag{4.4}$$

$$x_{t+1} = s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 \tag{4.5}$$

A dynamic portfolio policy is an investment sequence of functions

$$\pi = (\mu_0, \mu_1, \mu_2, ..., \mu_{T-1}) \tag{4.6}$$

$$= \left(\begin{bmatrix} \mu_0^1 \\ \mu_0^2 \\ \vdots \\ \mu_0^n \end{bmatrix}, \begin{bmatrix} \mu_1^1 \\ \mu_1^2 \\ \vdots \\ \mu_1^n \end{bmatrix}, ..., \begin{bmatrix} \mu_{T-1}^1 \\ \mu_{T-1}^2 \\ \vdots \\ \mu_{T-1}^n \end{bmatrix} \right) \tag{4.7}$$

where μ_t is the feedback control function that maps the wealth at the beginning of the period t, x_t, into a portfolio decision u_t such that

$$\begin{bmatrix} u_t^1 \\ u_t^2 \\ \vdots \\ u_t^n \end{bmatrix} = \begin{bmatrix} \mu_t^1(x_t) \\ \mu_t^2(x_t) \\ \vdots \\ \mu_t^n(x_t) \end{bmatrix} \tag{4.8}$$

A dynamic portfolio policy, π^*, is said to be efficient if there exists no other dynamic portfolio policy, π such that $E(x_T)|_\pi \geq E(x_T)|_{\pi^*}$ and $Var(x_T)|_\pi \leq Var(x_T)|_{\pi^*}$ with at least one strict inequality. For different degrees of risk aversion, we can vary the value of ν and solve a series of mathematical programming problem of $P(\nu)$. The efficient frontier can be obtained from the locus of the optimal solutions in the variance-mean space when ν varies. We can establish an equivalent formulation of a simpler form, $E(w)$, to generate efficient dynamic portfolio policies by varing parameter w.

$$\begin{aligned} E(w) &: \max\left(E(x_T) - wVar(x_T)\right) && (4.9) \\ s.t.\ x_{t+1} &= s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 && (4.10) \end{aligned}$$

where $w > 0$. Define $\Pi_1(w)$ to be the set of the optimal solution of problem $(E(w))$ with given w, i.e.

$$\Pi_1(w) = (\pi : \pi \text{ is a maximizer of } E(w)) \tag{4.11}$$

Lemma 1 *(Evertt's Theorem [32]) If $\pi(w)$ solves the Lagrangian problem $E(w)$, with $w \geq 0$, then $\pi(w)$ solves the orginal problem $P(\nu)$ where $Var(x_T)|\pi = \nu$ if $w > 0$ and $Var(x_T)|\pi < \nu$ if $w = 0$.*

4.3 Auxiliary Problem Formulation

Dynamic programming is one of the most universal and most powerful solution methodologies for stochastic sequential optimization problems with a separable structure. Problem $(E(w))$, however, is non-separable in the sense of dynamic programming and hence dynamic programming cannot be applied directly. As a result, we have to tackle the problem in an indirect way.

Define

$$\begin{aligned}
& \widetilde{U}\left(E\left(x_T\right), E\left(x_T^2\right)\right) \\
= & E\left(x_T\right) - wVar\left(x_T\right) \\
= & E\left(x_T\right) - w\left[E\left(x_T^2\right) - E^2\left(x_T\right)\right] \qquad (4.12)
\end{aligned}$$

It can be seen that $\widetilde{U}\left(E\left(x_T\right), E\left(x_T^2\right)\right)$ is a convex function of $E\left(x_T^2\right)$ and $E\left(x_T\right)$. We can construct the following auxiliary problem $\left(A\left(\lambda, w\right)\right)$,

$$\begin{aligned}
A\left(\lambda, w\right) &: \max\ E\left(-wx_T^2 + \lambda x_T\right) \\
s.t.\ x_{t+1} &= s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 \qquad (4.13)
\end{aligned}$$

Define $\Pi_2^*\left(\lambda, w\right)$ to be the set of the optimal solution of problem $\left(A\left(\lambda, w\right)\right)$ with given w and λ, i.e.

$$\Pi_2^*\left(\lambda, w\right) = \left(\pi : \pi \text{ is a maximizer of } A\left(\lambda, w\right)\right) \qquad (4.14)$$

Define

$$\begin{aligned}
d_\pi\left(\lambda, w\right) &= \frac{\partial \widetilde{U}}{\partial E\left(x_T\right)}|_\pi \\
&= 1 + 2wE\left(x_T\right)|_\pi \qquad (4.15)
\end{aligned}$$

The auxiliary problem $A\left(\lambda, w\right)$ is now a separable problem and can be solved by dynamic programming effectively. In the following, we are going to investigate the linkage between the solutions for $A\left(\lambda, w\right)$ and $E\left(w\right)$ so that we can solve $E\left(w\right)$ indirectly by solving the auxiliary problem of $A\left(\lambda, w\right)$. Most of these theorems have been reported in Li and Ng[33] .

Proporsition 1 *[34]If U is a first-order differentiable convex function, then*

$$U\left(\pi^{(2)}\right) \geq U\left(\pi^{(1)}\right) + \nabla U\left(\pi^{(1)}\right)\left(\pi^{(2)} - \pi^{(1)}\right) \tag{4.16}$$

Proof. By definition of convexity,

$$U\left(\alpha\pi^{(2)} + (1-\alpha)\pi^{(1)}\right) \leq \alpha U\left(\pi^{(2)}\right) + (1-\alpha)U\left(\pi^{(1)}\right) \tag{4.17}$$

for some $0 < \alpha \leq 1$. Rearrange the inequality, we have

$$\frac{U\left(\pi^{(1)} + \alpha\left(\pi^{(2)} - \pi^{(1)}\right)\right) - U\left(\pi^{(1)}\right)}{\alpha} \leq U\left(\pi^{(2)}\right) - U\left(\pi^{(1)}\right) \tag{4.18}$$

Letting $\alpha \to 0$, we have $\nabla U\left(\pi^{(1)}\right)\left(\pi^{(2)} - \pi^{(1)}\right) \leq U\left(\pi^{(2)}\right) - U\left(\pi^{(1)}\right)$ and thus $U\left(\pi^{(2)}\right) \geq U\left(\pi^{(1)}\right) + \nabla U\left(\pi^{(1)}\right)\left(\pi^{(2)} - \pi^{(1)}\right)$

□

Theorem 1 *For any $\pi^* \in \Pi_1^*(w^*)$, $\pi^* \in \Pi_2^*(d(\pi^*, w^*), w^*)$*

Proof. By contradiction, assume $\pi^* \notin \Pi_2^*(d(\pi^*, w^*), w^*)$, then there exists a π such that

$$[-w^*, d(\pi^*, w^*)]\left[\begin{array}{c} E(x_T^2) \\ E(x_T) \end{array}\right]\Bigg|_{\pi} > [-w^*, d(\pi^*, w^*)]\left[\begin{array}{c} E(x_T^2) \\ E(x_T) \end{array}\right]\Bigg|_{\pi^*} \tag{4.19}$$

Note that $\widetilde{U}(E(x_T^2), E(x_T))$ is a convex function of $E(x_T^2)$ and $E(x_T)$ and we have $\frac{\partial \widetilde{U}}{\partial E\left(x_T^2\right)} = -w$ and $\frac{\partial \widetilde{U}}{\partial E(x_T)} = d(\pi, w)$. According to the property of convexity

(Proposition 1), we can obtain the following,

$$\widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|\pi \geq \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|\pi^*$$
$$+\left[-w^*, d\left(\pi^*, w^*\right)\right]\left(\left.\begin{bmatrix} E\left(x_T^2\right) \\ E\left(x_T\right)\end{bmatrix}\right|_{\pi} - \left.\begin{bmatrix} E\left(x_T^2\right) \\ E\left(x_T\right)\end{bmatrix}\right|_{\pi^*}\right) \tag{4.20}$$

Combining the above two equations (4.19) and (4.20), we have

$$\widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|\pi > \widetilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right)|\pi^*$$

that contradicts the assumption of $\pi^* \in \Pi_1^*\left(w^*\right)$.

□

The implication of this theorem is that problem $E\left(w\right)$ can be embedded into a tractable auxiliary problem of $\left(A\left(\lambda, w\right)\right)$. The next question to be answered is under which condition a solution of $\left(A\left(\lambda, w\right)\right)$ constitutes an optimal dynamic portfolio policy of $E\left(w\right)$.

Theorem 2 *Assume that $\pi^* \in \Pi_2^*\left(\lambda^*, w^*\right)$, A necessary condition for $\pi^* \in \Pi_1^*\left(w^*\right)$ is $\lambda^* = 1 + 2w^* E\left(x_T\right)|_{\pi^*}$*

Proof. The solution set of problem $\left(A\left(\lambda, w^*\right)\right)$ can be parameterized by λ. Each point in the set $\Pi_2^*\left(\lambda, w^*\right)$ can be represented by $\left(E\left(x_T^2\left(\lambda, w^*\right)\right), E\left(x_T\left(\lambda, w^*\right)\right)\right)$. Since $\Pi_1^*\left(w^*\right) \subseteq \cup_\lambda \Pi_2^*\left(\lambda, w^*\right)$, problem $\left(E\left(w^*\right)\right)$ can be reduced in abstract to the following equivalent form,

$$\max_{\lambda} \quad \tilde{U}\left(E\left(x_T^2\left(\lambda, w^*\right)\right), E\left(x_T\left(\lambda, w^*\right)\right)\right) \tag{4.21}$$

$$= \max_{\lambda} \quad E\left(x_T\left(\lambda, w^*\right)\right) - w^*\left(E\left(x_T^2\left(\lambda, w^*\right)\right) - E^2\left(x_T\left(\lambda, w^*\right)\right)\right) \tag{4.22}$$

The first-order necessary condition for optimal λ^* is

$$-w^* \frac{\partial E\left(x_T^2\left(\lambda, w^*\right)\right)}{\partial \lambda} + \left[1 + 2w^* E\left(x_T\right)|_{\pi^*}\right] \frac{\partial E\left(x_T\left(\lambda, w^*\right)\right)}{\partial \lambda} = 0 \tag{4.23}$$

On the other hand, when $\pi^* \in \Pi_2^*\left(\lambda^*, w^*\right)$, we have from Reid and Citron[35],

$$-w^* \frac{\partial E\left(x_T^2\left(\lambda, w^*\right)\right)}{\partial \lambda} + \lambda^* \frac{\partial E\left(x_T\left(\lambda, w^*\right)\right)}{\partial \lambda} = 0$$

Thus, the vector $\left[-w^*, 1 + 2w^* E\left(x_T\right)|_{\pi^*}\right]$ is proportional to $\left[-w^*, \lambda^*\right]$. We must have $\lambda^* = 1 + 2w^* E\left(x_T\right)|_{\pi^*}$

□

In other words, the problem of $E(w)$ can be solved by embeding the problem into the auxiliary problem $\left(A\left(\lambda, w\right)\right)$ with a quadratic form of utility function in terms of $E\left(x_T^2\left(\lambda, w\right)\right)$ and $E\left(x_T\left(\lambda, w\right)\right)$. And the optimal solution is reached when the condition $\lambda^* = 1 + 2w^* E\left(x_T\right)|_{\pi^*}$ is satisfied.

The prominent feature of problem $\left(A\left(\lambda, w^*\right)\right)$ is that the problem is separable and of a quadratic objective function with linear constraints. The optimal solution for problem $(A(\lambda, w))$ can be derived analytically using dynamic programming. We start from period $T-1$,

$$\max J_{T-1}\left(x_{T-1}\right)$$
$$s.t. \ : \ x_T = s_{T-1}x_{T-1} + P'_{T-1}u_{T-1} \tag{4.24}$$

Problem (4.24) can be simplied to the following,

$$\begin{aligned} & \max J_{T-1}\left(x_{T-1}\right) \\ = \ & E\left(-wx_T^2 + \lambda x_T\right) \\ = \ & E\left(-w\left(s_{T-1}x_{T-1} + P'_{T-1}u_{T-1}\right)^2 + \lambda\left(s_{T-1}x_{T-1} + P'_{T-1}u_{T-1}\right)\right) \end{aligned} \tag{4.25}$$

$$\begin{aligned} = \ & -ws_{T-1}^2x_{T-1}^2 + \lambda s_{T-1}x_{T-1} + \left(\lambda - 2ws_{T-1}x_{T-1}\right)E\left(P'_{T-1}\right)u_{T-1} \\ & -wu'_{T-1}E\left(P_{T-1}P'_{T-1}\right)^{-1}u_{T-1} \end{aligned} \tag{4.26}$$

Differentiate $J_{T-1}\left(x_{T-1}\right)$ with respect to u_{T-1}, we have

$$\frac{\partial J_{T-1}\left(x_{T-1}\right)}{\partial u_{T-1}} = \left(\lambda - 2ws_{T-1}x_{T-1}\right)E\left(P_{T-1}\right) - 2wE\left(P_{T-1}P'_{T-1}\right)^{-1}u_{T-1} \tag{4.27}$$

Set $\frac{\partial J_{T-1}(x_{T-1})}{\partial u_{T-1}} = 0$, we can yield the optimal investment policy at time $T-1$,

$$u^*_{T-1} = E\left(P_{T-1}P'_{T-1}\right)^{-1}\left(\frac{\lambda}{2w} - s_{T-1}x_{T-1}\right)E\left(P_{T-1}\right) \tag{4.28}$$

Subsitituting u^*_{T-1} into $J_{T-1}\left(x_{T-1}\right)$, we have the optimal cost-to-go at given x_{T-1},

$$J^*_{T-1}\left(x_{T-1}\right)$$

$$
\begin{aligned}
&= -ws_{T-1}^2\left[1-E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right)\right]x_{T-1}^2 \\
&\quad +\lambda s_{T-1}\left[1-E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right)\right]x_{T-1} \\
&\quad +\frac{\lambda^2}{4w}E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right) \\
&= -w_{T-1}x_{T-1}^2+\lambda_{T-1}x_{T-1} \\
&\quad +\alpha_{T-1}E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right) \qquad (4.29)
\end{aligned}
$$

where

$$
\begin{aligned}
w_{T-1} &= ws_{T-1}^2\left[1-E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right)\right] && (4.30)\\
\lambda_{T-1} &= \lambda s_{T-1}\left[1-E\left(P_{T-1}'\right)E\left(P_{T-1}P_{T-1}'\right)^{-1}E\left(P_{T-1}\right)\right] && (4.31)\\
\alpha_{T-1} &= \frac{\lambda^2}{4w} && (4.32)
\end{aligned}
$$

The derived utility function has a similar form at period t, $0 \le t \le T-1$, to the original utility form at stage T. We can drive the optimal control and the optimal cost-to-go for given x_t at period t, $0 \le t \le T-2$, in a similar manner,

$$
\begin{aligned}
u_t^* &= E\left(P_tP_t'\right)^{-1}\left(\frac{\lambda_{t+1}}{2w_{t+1}}-s_tx_t\right)E\left(P_t\right) \\
&= E\left(P_tP_t'\right)^{-1}\left(\frac{\lambda}{2w\prod_{k=t+1}^{T-1}s_k}-s_tx_t\right)E\left(P_t\right) \qquad (4.33)
\end{aligned}
$$

$$
J_t^*\left(x_t\right) = -w_tx_t^2+\lambda_tx_t+\sum_{k=t}^{T-1}\alpha_kE\left(P_k'\right)E\left(P_kP_k'\right)^{-1}E\left(P_k\right) \qquad (4.34)
$$

where

$$
w_t = w_{t+1}s_t^2\left[1-E\left(P_t'\right)E\left(P_tP_t'\right)^{-1}E\left(P_t\right)\right] \qquad (4.35)
$$

$$
\begin{aligned}
\lambda_t &= \lambda_{t+1} s_t \left[1 - E(P_t') E(P_t P_t')^{-1} E(P_t)\right] && (4.36) \\
\alpha_t &= \frac{\lambda_{t+1}^2}{4 w_{t+1}} && (4.37)
\end{aligned}
$$

In summary, at any time t, the optimal control policy for $(A(\lambda, w))$ can be expressed as a linear function of x_t,

$$
u_t(x_t; \gamma) = -\mathbf{K}_t x_t + v_t(\gamma) \quad t = 0, 1, ..., T-1 \tag{4.38}
$$

where

$$
\begin{aligned}
\gamma &= \frac{\lambda}{w} && (4.39) \\
\mathbf{K}_t &= s_t E(P_t P_t')^{-1} E(P_t) \quad t = 0, 1, ..., T-1 && (4.40) \\
E(P_t P_t') &= Cov(e_t) + E(P_t) E(P_t') \quad t = 0, 1, ..., T-1 && (4.41)
\end{aligned}
$$

and

$$
\begin{aligned}
v_t(\gamma) &= \frac{\gamma}{2 \prod_{k=t+1}^{T-1} s_k} E(P_t P_t')^{-1} E(P_t) \\
t &= 0, 1, ..., T-2 && (4.42)
\end{aligned}
$$

with boundary condition

$$
v_{T-1}(\gamma) = \frac{\gamma}{2} E\left(P_{T-1} P_{T-1}'\right)^{-1} E(P_{T-1}) \tag{4.43}
$$

4.4 Efficient Frontier in Multi-period Portfolio Selection

Substituting (4.38) into the wealth dynamic (4.13) yields

$$\begin{aligned} x_{t+1}(\gamma) &= (s_t - P_t'\mathbf{K}_t)\, x_t(\gamma) + P_t' v_t(\gamma) \qquad (4.44) \\ t &= 0, 1, ..., T-1 \end{aligned}$$

Since it is assumed that P_t and $x_t(\gamma)$ are statistically independent, we have the following relationship if we take the expectation on both sides of equation (4.44).

$$\begin{aligned} E(x_{t+1}(\gamma)) &= s_t \left(1 - E(P_t)\, E(P_t P_t')^{-1} E(P_t)\right) E(x_t(\gamma)) \\ &\quad + \frac{\gamma}{2\prod_{k=t+1}^{T-1} s_k} E(P_t)\, E(P_t P_t')^{-1} E(P_t) \qquad (4.45) \end{aligned}$$

Denote

$$B_t = E(P_t)\, E(P_t P_t')^{-1} E(P_t) \qquad (4.46)$$

$$t = 0, 1, ..., T-1 \qquad (4.47)$$

Hence

$$E(x_{t+1}(\gamma)) = s_t (1 - B_t)\, E(x_t(\gamma)) + \frac{\gamma}{2\prod_{k=t+1}^{T-1} s_k} B_t \qquad (4.48)$$

Solving equation (4.48) recurssively, we can express $E(x_T(\gamma))$ explicitly in term of γ as

$$E(x_T(\gamma)) = \left(\prod_{t=0}^{T-1} s_t (1 - B_t)\right) x_0 + \sum_{t=0}^{T-1} \left(\prod_{k=t+1}^{T-1} (1 - B_k)\, B_t\right) \frac{\gamma}{2}$$

$$= \left(\prod_{t=0}^{T-1} s_t (1 - B_t)\right) x_0 + \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right) \frac{\gamma}{2} \qquad (4.49)$$

as the relationship of $\sum_{t=0}^{T-1} \left(\prod_{k=t}^{T-1} (1 - B_k) B_t\right) = \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right)$ is proven in the following proposition.

Proporsition 2 *For any length of planning horizon T, the relationship*

$$\sum_{t=0}^{T-1} \left(\prod_{k=t+1}^{T-1} (1 - B_k) B_t\right) = 1 - \prod_{t=0}^{T-1} (1 - B_t) \qquad (4.50)$$

holds.

Proof. We can prove the relationship $\sum_{t=0}^{T-1} \left(\prod_{k=t+1}^{T-1} (1 - B_k)\right) B_t = 1 - \prod_{t=0}^{T-1} (1 - B_t)$ by mathematical induction. If $k_1 > k_2$, we define

$$\prod_{t=k_1}^{k_2} (1 - B_t) = 1 \qquad (4.51)$$

For $T = 1$,

$$L.H.S. = B_0 \qquad (4.52)$$

$$R.H.S. = B_0 \qquad (4.53)$$

Hence, it is true for $T = 1$

Assume that the relationship holds for $T = \Re$, i.e.

$$\sum_{t=0}^{\Re-1} \left(\prod_{k=t+1}^{\Re-1} (1 - B_k)\right) B_t = 1 - \prod_{t=0}^{\Re-1} (1 - B_t) \qquad (4.54)$$

For $T = \Re + 1$,

$$\begin{aligned}
L.H.S. &= \sum_{t=0}^{\Re} \left(\prod_{k=t+1}^{\Re} (1 - B_k) \right) B_t \\
&= \sum_{t=0}^{\Re-1} \left(\prod_{k=t+1}^{\Re} (1 - B_k) \right) B_t + B_{\Re} \\
&= \sum_{t=0}^{\Re-1} \left(\prod_{k=t+1}^{\Re-1} (1 - B_k) (1 - B_{\Re}) \right) B_t + B_{\Re} \\
&= (1 - B_{\Re}) \sum_{t=0}^{\Re-1} \left(\prod_{k=t+1}^{\Re-1} (1 - B_k) \right) B_t + B_{\Re} \\
&= (1 - B_{\Re}) \left(1 - \prod_{t=0}^{\Re-1} (1 - B_t) \right) + B_{\Re} \\
&= 1 - \prod_{t=0}^{\Re} (1 - B_t) \\
&= R.H.S. \qquad (4.55)
\end{aligned}$$

Therefore, by mathematical induction, we can conclude that the relationship holds for any natural number T.

□

Similarly, we can take square on both sides of equation (4.44) to yield,

$$\begin{aligned}
x_{t+1}^2(\gamma) &= \left[s_t^2 - 2 s_t \mathbf{P}_t' \mathbf{K}_t + \mathbf{K}_t' \mathbf{P}_t \mathbf{P}_t' \mathbf{K}_t \right] x_t^2(\gamma) \\
&\quad + 2 (s_t - \mathbf{P}_t' \mathbf{K}_t) x_t(\gamma) \mathbf{P}_t' \mathbf{v}_t(\gamma) + \mathbf{v}_t(\gamma)' \mathbf{P}_t \mathbf{P}_t' \mathbf{v}_t(\gamma) \\
&\qquad t = 0, 1, \ldots, T-1 \qquad (4.56)
\end{aligned}$$

and then take expectation on both sides. The following equation is obtained after simplifications,

$$\begin{aligned} E\left(x_{t+1}^2(\gamma)\right) &= s_t^2\left(1-E(P_t)E(P_tP_t')^{-1}E(P_t)\right)E\left(x_t^2(\gamma)\right) \\ &\quad + \left(\frac{\gamma}{\prod_{k=t+1}^{T-1}s_k}\right)^2 E(P_t)E(P_tP_t')^{-1}E(P_t) \end{aligned} \tag{4.57}$$

Solving the above equation recursively, we can obtain the closed form of $E\left(x_T^2(\gamma)\right)$,

$$\begin{aligned} E\left(x_T^2(\gamma)\right) &= \left(\prod_{t=0}^{T-1}s_t^2(1-B_t)\right)x_0^2+\sum_{t=0}^{T-1}\left(\prod_{k=t+1}^{T-1}(1-B_k)B_t\right)\frac{\gamma^2}{4} \\ &= \left(\prod_{t=0}^{T-1}s_t^2(1-B_t)\right)x_0^2+\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)\frac{\gamma^2}{4} \end{aligned} \tag{4.58}$$

Consequently, we can express the variance of the final wealth explicitly

$$\begin{aligned} Var\left(x_T(\gamma)\right) &= E\left(x_T^2(\gamma)\right)-E^2\left(x_T(\gamma)\right) \\ &= \prod_{t=0}^{T-1}(1-B_t)\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)\left(\prod_{t=0}^{T-1}s_tx_0-\frac{\gamma}{2}\right)^2 \end{aligned} \tag{4.59}$$

All solutions for auxiliary problem $(A(\lambda,w))$ can be obtained parametrically in γ. The next step is to identify the condition of γ such that the solution is optimal to $E(w)$. Since both $E(x_T(\gamma))$ and $Var(x_T(\gamma))$ are in term of γ, we can therefore express $\tilde{U}(E(x_T(\gamma)),Var(x_T(\gamma)))$ in term of γ

$$\begin{aligned} &\tilde{U}\left(E(x_T(\gamma)),Var(x_T(\gamma))\right) \\ &= E(x_T(\gamma))-w\left(E\left(x_T^2(\gamma)\right)-E^2(x_T(\gamma))\right) \\ &= \left(\prod_{t=0}^{T-1}s_t(1-B_t)\right)x_0+\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)\frac{\gamma}{2} \\ &\quad -w\left(\prod_{t=0}^{T-1}(1-B_t)\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)\left(\prod_{t=0}^{T-1}s_tx_0-\frac{\gamma}{2}\right)^2\right) \end{aligned} \tag{4.60}$$

We take derivative of $\widetilde{U}$ with respect to γ and set it equal to zero. Optimal value of γ can be obtained when $\frac{d\widetilde{U}}{d\gamma} = 0$. i.e.,

$$\left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right)$$
$$\left(\frac{1}{2} - w\left(\frac{\gamma}{2} - \left(\prod_{t=0}^{T-1} s_t (1 - B_t)\right) x_0\right) - \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right)\frac{\gamma}{2}\right) = 0 \tag{4.61}$$

We thus have the analytical expression for optimal γ,

$$\gamma^* = 2\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{w\prod_{t=0}^{T-1}(1 - B_t)} \tag{4.62}$$

The corresponding $v_t(\gamma)$ term in u_t^* now becomes

$$v_t(\gamma^*) = \frac{\left(\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{2w\prod_{t=0}^{T-1}(1-B_t)}\right)}{\prod_{k=t+1}^{T-1} s_k} E\left(P_t P_t'\right)^{-1} E\left(P_t\right) \tag{4.63}$$

The optimal investment at the t-th period is therefore

$$\begin{aligned} u_t^*(x_t; w) \;\; &= \;\; -s_t E\left(P_t P_t'\right)^{-1} E\left(P_t\right) x_t &(4.64) \\ &+ \left(\prod_{t=0}^{t} s_k x_0 + \frac{1}{2w\prod_{k=t+1}^{T-1} s_k \prod_{t=0}^{T-1}(1 - B_t)}\right) E\left(P_t P_t'\right)^{-1} E\left(P_t\right) \end{aligned}$$

The optimal expected value and the variance of the final wealth for $E(w)$ can be parameterized in w as

$$E\left(x_T(w)\right) = \prod_{t=0}^{T-1} s_{tt} x_0 + \frac{1 - \prod_{t=0}^{T-1}(1 - B_t)}{2w\prod_{t=0}^{T-1}(1 - B_t)} \tag{4.65}$$

and

$$Var\left(x_T(w)\right) = \frac{1 - \prod_{t=0}^{T-1}(1 - B_t)}{4w^2\prod_{t=0}^{T-1}(1 - B_t)} \tag{4.66}$$

The mean-variance efficient frontier thus can be obtained by eliminating the parameter w in $E(x_T)$ and $Var(x_T)$. From equation (4.65), we have

$$E(x_T(w)) - \prod_{t=0}^{T-1} s_{tt} x_0 = \frac{1 - \prod_{t=0}^{T-1}(1 - B_t)}{2w \prod_{t=0}^{T-1}(1 - B_t)} \tag{4.67}$$

From equation (4.66), we have

$$Var(x_T) = \left(\frac{\prod_{t=0}^{T-1}(1 - B_t)}{1 - \prod_{t=0}^{T-1}(1 - B_t)}\right)\left(\frac{1 - \prod_{t=0}^{T-1}(1 - B_t)}{2w \prod_{t=0}^{T-1}(1 - B_t)}\right)^2 \tag{4.68}$$

Combing equations (4.67) and (4.68) yields the analytical expression of the efficient frontier,

$$Var(x_T) = \frac{\prod_{t=0}^{T-1}(1 - B_t)}{\left(1 - \prod_{t=0}^{T-1}(1 - B_t)\right)}\left(\prod_{t=0}^{T-1} s_t x_0 - E(x_T)\right)^2 \tag{4.69}$$

4.5 Obseravtions

We now have completely derived the analytical form for the efficient frontier in multi-period portfolio selection. For an investor with a specific value of w, we also have a closed-form solution for his/her optimal investment policy, the expected terminal wealth and the corresponding risk.

1. The derived optimal portfolio policy u_t is of a linear form. It should notice that the optimal portfolio policy is not completely myopic as it takes some information beyond period t into account. The first component in u_t is proportional to the current wealth and it is independent of neither λ nor w

while the second component is a constant dependent on γ, or $\frac{\lambda}{w}$. In addition, we can see that the investment is not proportional to the wealth.

2. We have $\frac{dE\left(x_T^2(\gamma)\right)}{d\gamma}/\frac{dE(x_T(\gamma))}{d\gamma} = \gamma$. Therefore, $\frac{\partial E\left(x_T^2(\gamma)\right)}{\partial E(x_T(\gamma))} = \gamma$. The value γ can be viewed as a measure of the increasement of $E\left(x_T^2\left(\gamma\right)\right)$ for every unit of increasement of $E\left(x_T\left(\gamma\right)\right)$.

3. The expected final wealth $E\left(x_T\left(\gamma\right)\right)$ is a linear function of γ whereas the variance term of the final wealth, $Var\left(x_T\left(\gamma\right)\right)$, is of a quadratic form of γ.

4. The variance term $Var\left(x_T\left(\gamma\right)\right)$ is a prefect square term. And thus there exists a minimum-risk portfolio of zero variance. The portfolio can be obtained when $\gamma = 2\prod_{k=t}^{T-1} s_k x_0$ (from eqution (4.59)) or $E\left(x_T\right) = \prod_{t=0}^{T-1} s_t x_0$ (from the efficient frontier (4.69)) . In this case, the investor has $v_t\left(\gamma\right) = s_t E\left(P_t P_t'\right)^{-1} E\left(P_t\right) x_t$ and $u_t = 0$. i.e., he/she invests all his/her wealth in the risk-free asset for all time periods.

5. The value of B_t can be proven to be within the range $[0,1]$ for $t = 0, 1, ..., T-1$. Let us consider a simple case with only two risky assets, A and B. As a matter of fact, the risky assets, A or B or both, can be a portfolio of other assets. Hence the following result can be generalized for arbitrary portfolio

containing any number of individual securities. At an arbitrary time t, define

$$E(P_t) = \begin{pmatrix} P_A \\ P_B \end{pmatrix} \tag{4.70}$$

$$Cov(P_A, P_B) = \begin{bmatrix} \sigma_{AA} & \sigma_{AB} \\ \sigma_{BA} & \sigma_{BB} \end{bmatrix} \tag{4.71}$$

We assumd that the expected risk premium for the two risky assets,P_A, P_B are positive. Hence

$$B = \begin{pmatrix} P_A & P_B \end{pmatrix} \begin{bmatrix} \sigma_{AA} + P_A^2 & \sigma_{AB} + P_A P_B \\ \sigma_{AB} + P_A P_B & \sigma_{BB} + P_B^2 \end{bmatrix}^{-1} \begin{pmatrix} P_A \\ P_B \end{pmatrix} \tag{4.72}$$

$$= \frac{\begin{pmatrix} P_A & P_B \end{pmatrix} \begin{bmatrix} \sigma_{BB} + P_B^2 & -(\sigma_{AB} + P_A P_B) \\ -(\sigma_{AB} + P_A P_B) & \sigma_{AA} + P_A^2 \end{bmatrix} \begin{pmatrix} P_A \\ P_B \end{pmatrix}}{\det(E(PP'))} \tag{4.73}$$

$$= \frac{[P_A^2 \sigma_{BB} - 2\sigma_{AB} P_A P_B + P_B^2 \sigma_{AA}]}{\det(E(PP'))} \tag{4.74}$$

Note that

$$\det(E(PP')) = \left(1 - \rho_{AB}^2\right) \sigma_A^2 \sigma_B^2 + \left[P_A^2 \sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B P_A P_B + P_B^2 \sigma_A^2\right] \tag{4.75}$$

where ρ_{AB} is the coefficient of correlation between security A and securty B,

$$\rho_{AB} = \frac{\sigma_{AB}}{\sigma_A \sigma_B} \tag{4.76}$$

Define

$$\Theta = \left[P_A^2\sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B P_A P_B + P_B^2\sigma_A^2\right] \tag{4.77}$$

It can be shown that $\Theta \geq 0$ provided that P_A and P_B are positive,

$$\begin{aligned}
\Theta &= \left[P_A^2\sigma_B^2 - 2\rho_{AB}\sigma_A\sigma_B P_A P_B + P_B^2\sigma_A^2\right] \\
&= \left[P_A^2\sigma_B^2 - 2\sigma_A\sigma_B P_A P_B + P_B^2\sigma_A^2\right] + 2\left(1-\rho_{AB}\right)\sigma_A\sigma_B P_A P_B \\
&= \left[P_A\sigma_B - P_B\sigma_A\right]^2 + 2\left(1-\rho_{AB}\right)\sigma_A\sigma_B P_A P_B \qquad (4.78) \\
&\geq 0
\end{aligned}$$

as ρ_{AB} is always less than one. From the following expression,

$$B = \frac{\Theta}{\left(1-\rho_{AB}^2\right)\sigma_A^2\sigma_B^2 + \Theta} \tag{4.79}$$

since $\rho_{AB}^2 \leq 1$, , and thus we have $0 \leq B \leq 1$.

6. Continue the last observartion. We are interested in the situation where $\rho_{AB} = 1$ or $\rho_{AB} = -1$. If the two risky assets are of positive perfect correlation, i.e. $\rho_{AB} = 1$, then $B = 1$. If $\rho_{AB} = 1$, we can express P_A in terms of P_B as

$$\begin{aligned}
P_A &= \Lambda P_B \quad \text{for some constan } \Lambda \qquad (4.80) \\
\sigma_A &= \Lambda\sigma_B \qquad (4.81)
\end{aligned}$$

Any investor can thus construct a portfolio by longing a unit of asset A and shorting Λ units of asset B simultaneously as there is no limitation on u_t.

The risk for this portfolio ($\sigma_p = \sigma_A - \Lambda\sigma_B$) will therefore be 0. Similarly if $\rho_{AB} = -1$, then $B = 1$. One can have risk-free portfolio by simutanouly buying two assets with a proper ratio.

7. In both cases of $\rho_{AB} = 1$ and $\rho_{AB} = -1$, the risk $Var(x_T)$ in equation (4.69) is equal to zero. We have risk-free portfolio even we invest in risky market. In other words, the risk is reduced by diversification. It should be noticed that, in the situations where $\rho_{AB} = 1$ or $\rho_{AB} = -1$, there is only one efficient solution with $w = 0$, $\gamma = \infty$ and $E(x_T(\gamma)) = \infty$.

8. Since $0 \leq B_t \leq 1$, the term $\frac{\prod_{t=0}^{T-1}(1-B_t)}{\left(1-\prod_{t=0}^{T-1}(1-B_t)\right)}$ in efficient frontier is positive. The efficient frontier is a concave curve in the variance-mean space similar to that in the single period case.

4.6 Solution Algorithm for Problem $E(w)$

The problem of $E(w)$ can now be solved analytically by the following algorithm.

Algorithm 4.1 Solution Algorithm for Problem $E(w)$

1 Obtain $s_t, E(e_t)$, and $Cov(e_t)$, x_0

2 **for** $t = 1$ to $T-1$

3 $\quad E(P_t) = E(e_t - s_t)$,

4 $\quad E(P_t P_t') = Cov(e) + E(P_t)E(P_t')$, and $B_t = E(P_t)E(P_t P_t')^{-1}E(P_t)$

$\quad \mathbf{K}_t = s_t E\left(P_t P_t'\right)^{-1} E\left(P_t\right)$

end for

Given w

$\gamma^* = 2\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{w\prod_{t=0}^{T-1}(1-B_t)}$

$E\left(x_T(w)\right) = \prod_{t=0}^{T-1} s_{tt} x_0 + \frac{1-\prod_{t=0}^{T-1}(1-B_t)}{2w\prod_{t=0}^{T-1}(1-B_t)}$ and $Var\left(x_T(w)\right) = \frac{1-\prod_{t=0}^{T-1}(1-B_t)}{4w^2\prod_{t=0}^{T-1}(1-B_t)}$

$v_{T-1}\left(\gamma^*\right) = \frac{\gamma^*}{2} E\left(P_{T-1}P_{T-1}'\right)^{-1} E\left(P_{T-1}\right)$

for $t = 0$ to $T-2$,

$\quad v_t\left(\gamma^*\right) = \frac{\gamma^*}{2\prod_{k=t+1}^{T-1} s_k} E\left(P_t P_t'\right)^{-1} E\left(P_t\right)$

end for

4.7 Illstrative Examples

To illustrate our proposed framework and solution algorithm (Algorithm 4.1), we carry out some case studies by using data from a standard textbook of investment by Sharp *et al* [20].

Example 1 *An investor has* 1 *unit of wealth at the very beginning of planning horizon* $(t = 0)$. *He has to find a best policy for the following four periods* $(T = 4)$ *by allocating his wealth into a safety asset paying rate of return* S *and three risk securities namely A, B and C. The process is assumed to be stationary such that* $s_t = S$, *and* $E\left(P_t'\right)E\left(P_tP_t'\right)^{-1}E\left(P_t\right) = B$, $t = 0, 1, 2, 3$. *It is assumed that the returns are following multivariate normal distribution. The mean rate of return* *of the securities are* $E\left(e_t^A\right) = 1.162$, $E\left(e_t^B\right) = 1.246$ *and* $E\left(e_t^C\right) = 1.228$ $t = 0, 1, 2, 3$. *The variance-covariance matrix* $Cov\left(e_t\right) = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}$, $t = 0, 1, 2, 3$.

The safety rate of return S_t *is set to be* 1.04 *and* $P_t = \left[P_t^A, P_t^B, P_t^C\right]$ *for each stock can then be calculated as*

$$P_t = \begin{bmatrix} P_t^A \\ P_t^B \\ P_t^C \end{bmatrix} = \begin{bmatrix} 0.1220 \\ 0.2060 \\ 0.1880 \end{bmatrix} \quad t = 0, 1, 2, 3 \tag{4.82}$$

The value of B_t is 0.593817, $t = 0, 1, 2, 3$. One can have the following results immediately from the equations derived in the pervious chapters.

$$E\left(x_T\right) = S^T\left(1 - B\right)^T x_0 + \left[1 - \left(1 - B\right)^T\right]\frac{\gamma}{2} \tag{4.83}$$

$$= 0.031843 + 0.4864\gamma \tag{4.84}$$

$$E\left(x_T^2\right) = S^{2T}\left(1 - B\right)^T x_0 + \left[1 - \left(1 - B\right)^T\right]\frac{\gamma^2}{4} \tag{4.85}$$

$$= 0.03725 + 0.2432\gamma^2 \tag{4.86}$$

and hence,

$$Var\left(x_T\right) = \left[1 - \left(1 - B\right)^T\right]\left(1 - B\right)^T\left(S^T x_0 - \frac{\gamma}{2}\right)^2 \tag{4.87}$$

$$= 0.0066\gamma^2 - 0.0310\gamma + 0.0365 \tag{4.88}$$

The optimal policy $u(t) = -\mathbf{K}_t x_t + v_t(\gamma)$ can be divided into two components. $\mathbf{K}_t$ is a constant vector for any w and does not change with time t in a stationary model. $v_t(\gamma)$, however, is changing for different value of w and at different time t. In this example, $\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}$, $t = 0, 1, 2, 3$, while $v_t(\gamma)$ for different values of w are summaried in tables (4.1)-(4.12).

The amount of investment in riskfree asset at period t is $(x_t - \sum u_t^i)$. We can then have the following.

$$
\begin{aligned}
u_t^{riskfree} &= x_t - \sum u_t^i && (4.89) \\
&= x_t + \left(\sum K_t^i\right) x_t - \sum v_t^i(\gamma) && (4.90) \\
&= -K_t^{riskfree} + v_t^{riskfree}(\gamma) && (4.91)
\end{aligned}
$$

where

$$K_t^{riskfree} = -\left(1 + \sum K_t^i\right) \tag{4.92}$$

and

$$v_t^{riskfree}(\gamma) = -\sum v_t^i(\gamma) \tag{4.93}$$

The proportion in investment of risky asset i is $\frac{-K_t^i x_t + v_t^i(\gamma)}{x_t}$. Since $v_t^i(\gamma)$ is independent of x_t and K_t is a constant vector over time, a larger value of $v_t^i(\gamma)$ will induce a relatively larger proportion of wealth at period t invested in security i.

We have observed from the results that the value of $v_t^i(\gamma)$ in any risky asset i decreases as the vaule of w increases while the value of $v_t^{riskfree}(\gamma)$ increases. This confirms that an investor is more unwilling to invest in risky assets when he/she is more risk-averse.

As we can see, from the results that the values of $v_t^i(\gamma)$, $\forall t$, are increasing over time, we can find that the investor increases the relative proportion of his investment in all the risky assts when the time is getting closer to the planning horizon . We guess that when the time is approaching to the end of planning horizon T, the future becomes clearer and the investor becomes less cautions. In other words, in our framework, we have also considered the "time effect" in Mossin[6] where a qraduatic utility terminal wealth maximization is investigated.

As part of future information, prediction of rate of returns riskfree asset in future, are used to determinate the optimal investment policy, we can conclude that the optimal policy is partially non-myopic.

$w = 0.001$				
$\gamma = 36740.2344$				
t	0	1	2	3
A	0.6288e04	0.6539e04	0.6801e04	0.7073e04
B	1.0200e04	1.0608e04	1.1033e04	1.1474e04
C	3.6326e04	3.7779e04	3.9290e04	4.0862e04

Table 4.1: $v_t(\gamma)$ for Example 1 when $w = 0.001$

$w = 0.01$				
$\gamma = 3676.1292$				
t	0	1	2	3
A	0.6291e03	0.6543e03	0.6805e03	0.7077e03
B	1.0206e03	1.0614e03	1.1039e03	1.1481e03
C	3.6347e03	3.7801e03	3.9313e03	4.0885e03

Table 4.2: $v_t(\gamma)$ for Example 1 when $w = 0.01$

$w = 0.02$				
$\gamma = 1839.2345$				
t	0	1	2	3
A	0.3148e03	0.3274e03	0.3404e03	0.3541e03
B	0.5106e03	0.5311e03	0.5523e03	0.5744e03
C	1.8185e03	1.8912e03	1.9669e03	2.0456e03

Table 4.3: $v_t(\gamma)$ for Example 1 when $w = 0.02$

$w = 0.05$				
$\gamma = 737.0976$				
t	0	1	2	3
A	126.1444	131.1902	136.4378	141.8953
B	204.6423	212.8279	221.3411	230.1947
C	728.7846	757.9360	757.9360	819.7836

Table 4.4: $v_t(\gamma)$ for Example 1 when $w = 0.05$

$w = 0.1$				
$\gamma = 369.7187$				
t	0	1	2	3
A	63.2724	65.8033	68.4354	71.1729
B	102.6459	106.7518	111.0218	115.4627
C	365.5490	380.1709	395.3778	411.1929

Table 4.5: $v_t(\gamma)$ for Example 1 when $w = 0.1$

$w = 1$				
$\gamma = 39.0776$				
t	0	1	2	3
A	6.876	6.9551	7.2333	7.5227
B	10.8492	11.2832	11.7345	12.2039
C	38.6369	40.1824	41.7897	43.4613

Table 4.6: $v_t(\gamma)$ for Example 1 when $w = 1$

$w = 2$				
$\gamma = 20.7087$				
t	0	1	2	3
A	3.5440	3.6858	3.8332	3.9865
B	5.7494	5.9794	6.2185	6.4673
C	20.4751	21.2941	22.1459	23.0317

Table 4.7: $v_t(\gamma)$ for Example 1 when $w = 2$

$w = 3$				
$\gamma = 14.5857$				
t	0	1	2	3
A	2.4961	2.5960	2.6998	2.8078
B	4.0495	4.2114	4.3799	4.5551
C	14.4212	14.9980	15.5980	16.2219

Table 4.8: $v_t(\gamma)$ for Example 1 when $w = 3$

$w = 4$				
$\gamma = 11.5242$				
t	0	1	2	3
A	1.9722	2.0511	2.1331	2.2185
B	3.1995	3.3275	3.4606	3.5990
C	11.3942	11.8500	12.3240	12.8169

Table 4.9: $v_t(\gamma)$ for Example 1 when $w = 4$

$w = 5$				
$\gamma = 9.6873$				
t	0	1	2	3
A	1.6579	1.7242	1.7931	1.8649
B	2.6895	2.7971	2.9090	3.0253
C	9.5780	9.9612	10.3596	10.7740

Table 4.10: $v_t(\gamma)$ for Example 1 when $w = 5$

4.8 Verification with Single-period Efficient Frontier

To further verify our results, we apply our solution to the single period portfolio problem by setting $T = 1$ and check the result with the well established mean-variance efficient frontier in the literature. Single-period efficient frontier involving a risk-less asset was derived by Merton[23] and can be expressed as follows using our notation,

$$\left(E\left(x_1\right) - s\right)^2 = Var\left(x_1\right)\left(\mathcal{C}s^2 - 2\mathcal{A}s + \mathcal{B}\right) \tag{4.94}$$

where

$$\mathcal{A} = \mathbf{1}'\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\mathbf{e}\right) \tag{4.95}$$

$$\mathcal{B} = E\left(\mathbf{e}'\right)\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\mathbf{e}\right) \tag{4.96}$$

$$\mathcal{C} = \mathbf{1}'\left(Cov\left(\mathbf{e}\right)\right)^{-1}\mathbf{1} \tag{4.97}$$

$$\mathbf{1} = (1, 1, ..., 1)' \in R^n \tag{4.98}$$

We can rewrite $\mathcal{C}s^2 - 2\mathcal{A}s + \mathcal{B}$ as,

$$\begin{aligned}
& \mathcal{C}s^2 - 2\mathcal{A}s + \mathcal{B} \\
= \; & \left(\mathbf{1}'\left(Cov\left(\mathbf{e}\right)\right)^{-1}\mathbf{1}\right)s^2 - 2\left(\mathbf{1}'\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\mathbf{e}\right)\right)s + E\left(\mathbf{e}'\right)\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\mathbf{e}\right) \\
= \; & E\left(\left(\mathbf{e}-\mathbf{s}\right)'\right)\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\left(\mathbf{e}-\mathbf{s}\right)\right) \\
= \; & E\left(\mathbf{P}'\right)\left(Cov\left(\mathbf{e}\right)\right)^{-1}E\left(\mathbf{P}\right)
\end{aligned} \tag{4.99}$$

$w = 10$				
$\gamma = 6.0135$				
t	0	1	2	3
A	1.0291	1.0703	1.1131	1.1576
B	1.6695	1.7363	1.8058	1.8780
C	5.9457	6.1835	6.4309	6.6881

Table 4.11: $v_t(\gamma)$ for Example 1 when $w = 10$

$w = 100$				
$\gamma = 2.7071$				
t	0	1	2	3
A	0.4633	0.4818	0.5011	0.5211
B	0.7516	0.7816	0.8129	0.8454
C	2.6766	2.7836	2.8950	3.0108

Table 4.12: $v_t(\gamma)$ for Example 1 when $w = 100$

where $\mathbf{s} = (s, s, ..., s)' \in R^n$.

Define a matrix as

$$\begin{bmatrix} E(\mathbf{PP}') & E(\mathbf{P}) \\ E(\mathbf{P}') & 1 \end{bmatrix}$$

and then we can apply directly the formula of inversion of matrix by partitioning [36] to obtain the following relationship.

$$\begin{aligned} & 1^{-1} + 1^{-1}E(\mathbf{P}')\left(E(\mathbf{PP}') - E(\mathbf{P})1^{-1}E(\mathbf{P}')\right)^{-1}E(\mathbf{P})1^{-1} \\ = & \left(1 - E(\mathbf{P}')E(\mathbf{PP}')^{-1}E(\mathbf{P})\right)^{-1} \end{aligned} \tag{4.100}$$

The left-hand side can be simplified as $1 + E(\mathbf{P}')(Cov(\mathbf{e}))^{-1}E(\mathbf{P})$. Hence

$$\begin{aligned} & \mathcal{C}s^2 - 2\mathcal{A}s + \mathcal{B} \\ = & E(\mathbf{P}')(Cov(\mathbf{e}))^{-1}E(\mathbf{P}) \\ = & \left(1 - E(\mathbf{P}')E(\mathbf{PP}')^{-1}E(\mathbf{P})\right)^{-1} - 1 \\ = & \frac{1}{(1-B)} - 1 \\ = & \frac{B}{(1-B)} \end{aligned} \tag{4.101}$$

Substituting back into (4.94), we have

$$(E(x_1) - s)^2 = Var(x_1)\left(\frac{B}{(1-B)}\right)$$

i.e.,

$$Var(x_1) = \frac{1-B}{B}(E(x_1) - s)^2$$

which is the same as the efficient frontier derived by our dynamic portfolio framework when setting $x_0 = 1$ and $T = 1$.

4.9 Generalization to Cases with Nonlinear Utility Function of $E(x_T)$ and $Var(x_T)$

The parametric framework derived above for a linear utility of $E(x_T)$ and $Var(x_T)$ can be adopted to handle a more general class of nonlinear utility functions of $E(x_T)$ and $Var(x_T)$. The objective of an investor is always to maximize his expected final wealth $E(x_T)$ while minimizing the variance of the terminal wealth $Var(x_T)$. In other words, a utility function of a rational investor, $U(E(x_T), Var(x_T))$, should satisfy the following,

$$\frac{\partial U(E(x_T), Var(x_T))}{\partial E(x_T)} > 0 \tag{4.102}$$

$$\frac{\partial U(E(x_T), Var(x_T))}{\partial Var(x_T)} < 0 \tag{4.103}$$

A general model for dynamic portfolio selection is proposed as follows,

$$P(U): \quad \max_{\pi} U(E(x_T), Var(x_T)) \tag{4.104}$$

$$s.t.\ x_{t+1} = s_t x_t + P_t' u_t \quad t = 0, 1, 2, ..., T-1 \tag{4.105}$$

Define Π^* to be the set of optimal solutions of problem $P(U)$, i.e.

$$\Pi^* = (\pi | \pi \text{ is the maximizer of } P(U)) \tag{4.106}$$

The objective function of $P(U)$ can be linear or nonlinear. The linear utility function, $\widetilde{U}$, in the previous study can be treated as a special case of the general utility form of $E(x_T)$ and $Var(x_T)$ presented here. To generate the optimal solution for $P(U)$ by using the parametric framework, we should first investigate the relationship between the solution of $E(w)$ and $P(U)$.

Theorem 3 *If $\pi^* \in \Pi^*$, then there exists a $w \geq 0$ such that $\pi^* \in \Pi_1^*(w^*)$.*

Proof. Since U is an increasing function of $E(x_T)$ and a decreasing function of Var(x_T), the optimal solution of problem $P(U)$ must be on the mean-variance efficient frontier in the $(E(x_T), Var(x_T))$ space. Essentially, we can see that the efficient frontier for problem $E(w)$ is a concave function. Thus, there exists supporting lines everywhere on the efficient frontier.

In other words, every efficient solution, including $\pi^* \in \Pi^*$, can be generated by $(E(w))$ with a $w \geq 0$.

□

Theorem 4 *Assume $\pi^* \in \Pi_1^*(w^*)$. A necessary condition for $\pi^* \in \Pi^*$ is $w^* = -\left[\frac{\frac{\partial U}{\partial Var(x_T(w^*))}}{\frac{\partial U}{\partial E(x_T(w^*))}}\right]$.*

Proof. The efficient frontier, as we have discussed in the pervious section, is parameterized by w in the space of $(E(x_T), Var(x_T))$. We can represent any point on the efficient frontier by $(E(x_T(w)), Var(x_T(w)))$. Since $\Pi^* \subseteq \cup_w \Pi_1^*(w)$. Problem $(P(U))$ can be reduced in abstract to the following equivalent form,

$$\max_{w \geq 0} U(E(x_T(w)), Var(x_T(w)))$$

A first-order necessary condition for optimum w^* is

$$\frac{\partial U}{\partial E(x_T)}|_{\pi^*} \frac{\partial E(x_T(w^*))}{\partial w} + \frac{\partial U}{\partial Var(x_T(w^*))}|_{\pi^*} \frac{\partial Var(x_T(w^*))}{\partial w} = 0 \tag{4.107}$$

On the other hand, when $\pi^* \in \Pi_1^*(w^*)$, we have from Reid and Citron[35],

$$\frac{\partial E(x_T(w^*))}{\partial w} - w^* \frac{\partial Var(x_T(w^*))}{\partial w} = 0 \tag{4.108}$$

Thus, vector $\left[\frac{\partial U}{\partial E(x_T)}, \frac{\partial U}{\partial Var(x_T(w^*))}\right]\Big|_{\pi^*}$ is proportional to $[1, -w^*]$. We must have $w^* = -\left[\frac{\frac{\partial U}{\partial Var(x_T)}}{\frac{\partial U}{\partial E(x_T)}}\right]\Bigg|_{\pi^*}$.

□

Theorem 5 *Assume* $\pi^* \in \Pi_2^*(\lambda^*, w^*)$. *Necessary conditions for* $\pi^* \in \Pi^*$ *are* $w^* = -\left[\frac{\frac{\partial U}{\partial Var(x_T)}}{\frac{\partial U}{\partial E(x_T)}}\right]\Bigg|_{\pi^*}$ *and* $\lambda^* = 1 + 2w^* E(x_T)|_{\pi^*}$

Proof. This theorem can be easily proven using Theorems 2 and 4.

□

These theorems imply that the solution of $P(U)$ can be obtained by solving problem $(A(\lambda, w))$. In other words, we can embed the problem $P(U)$ into the auxiliary problem $(A(\lambda, w))$. Recall that the optimal parameter γ^* is obtained from $\frac{du}{d\gamma} = 0$. For problem $P(U)$, the derivative of $\frac{du}{d\gamma}$ can be obtained by applying the chain-rule,

$$\begin{aligned} \frac{dU}{d\gamma} &= \left[\frac{\partial U}{\partial E(x_T)} - 2E(x_T)\frac{\partial U}{\partial Var(x_T)}\right]\frac{dE(x_T)}{d\gamma} \\ &\quad + \frac{\partial U}{\partial Var(x_T)}\frac{dE(x_T^2)}{d\gamma} \end{aligned} \tag{4.109}$$

where

$$\frac{dE(x_T(\gamma))}{d\gamma} = \frac{1}{2}\left(1 - \prod_{t=0}^{T-1}(1 - B_t)\right)$$

and

$$\frac{dE(x_T^2(\gamma))}{d\gamma} = \frac{1}{2}\left(1 - \prod_{t=0}^{T-1}(1 - B_t)\right)\gamma$$

Since we have known that $\frac{dE(x_T^2(\gamma))}{d\gamma} / \frac{dE(x_T(\gamma))}{d\gamma} = \gamma$, we can obtain the following by setting $\frac{dU}{d\gamma} = 0$.

$$\gamma^* = 2E(x_T) - \frac{\frac{\partial U}{\partial E(x_T(\gamma^*))}}{\frac{\partial U}{\partial Var(x_T(\gamma^*))}} \tag{4.110}$$

However, both $E(x_T)$ and $Var(x_T)$ are depending on γ. Such coupling relationships make the above equation useful only in checking the result. We have to search for optimal γ by some numerical procedure. Because the gradient of U with respect to γ, $\frac{du}{d\gamma}$, is obtainable, we can make use of the gradient-dependent method such as gradient-search method or the false position method[34]. The general algorithm for solving problem $P(U)$ is summarized as following.

Algorithm 4.2 Solution Algorithm for Problem $P(U)$ with a Riskfree Asset

1 Obtain $s_t, E(e_t)$, and $Cov(e_t)$, x_0

for $t = 1$ to $T-1$

$\quad E(P_t) = E(e_t - s_t)$,

$\quad E(P_tP_t') = Cov(e) + E(P_t)E(P_t')$, and $B_t = E(P_t')E(P_tP_t')^{-1}E(P_t)$

$\quad \mathbf{K}_t = s_tE(P_tP_t')^{-1}E(P_t)$

end for

Set iteration number $j = 0$,

Choose an initial value of $\gamma, \gamma^{(0)}$.

Select a tolerance level ε.

while $\left|\frac{dU}{d\gamma}|_{\gamma}^{(i)}\right| \leq \varepsilon$

$\quad$ Calculate $E\left(x_T\left(\gamma^{(j)}\right)\right)$ using equation (5.45), $E\left(x_T^2\left(\gamma^{(j)}\right)\right)$ using equation (5.46) and hence $\frac{dU}{d\gamma}|_{\gamma}^{(i)}$ using equation (4.109).

$\quad j = j + 1$,

$\quad$ update $\gamma^{(j)}$ by gradient search, false position method or any method available

end while

set $\gamma^* = \gamma^{(j)}$,

$v_{T-1}(\gamma^*) = \frac{\gamma^*}{2}E\left(P_{T-1}P_{T-1}'\right)^{-1}E(P_{T-1})$

for $t = 0$ to $T-2$,

$\quad v_t(\gamma^*) = \frac{\gamma^*}{2\prod_{k=t+1}^{T-1}s_k}E(P_tP_t')^{-1}E(P_t)$

end for

The following two problems illustrate the procedures of Algorithm4.2.

Example 2 *Let us reconsider the Example 1. The investor is now seeking for* *optimal investment policy so as to maximize his utility described by*

$$U\left(E\left(x_T\right), Var\left(x_T\right)\right) = E\left(x_T\right)^2 - \exp\left(Var\left(x_T\right)\right) \tag{4.111}$$

We have the following from (4.109)

$$\frac{dU}{d\gamma} = 2E\left(x_T\right)\left(1 + \exp\left(Var\left(x_T\right)\right)\right)\frac{\partial E\left(x_T\right)}{\partial\gamma} - \exp\left(Var\left(x_T\right)\right)\frac{\partial E\left(x_T^2\right)}{\partial\gamma} \tag{4.112}$$

We adopt the false-position method to search for optimal γ with error tolence being 0.00001. The inital boundary values of γ are set to be 20 and 30. The Table 4.13 summarises the results in the iteration process.

Eventaully, the optimal value of γ is obtained to be 25.8964869. The value of $\mathbf{K}_t$ is independent of time t in this stationary model and $\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}$, $t = 0, 1, 2, 3$. The corresponding $v_t(\gamma)$ for $t = 0, 1, 2, 3$, are given in Table 4.14.

Example 3 *Let us reconsider the Example 1 again. The investor is now seeking for optimal investment policy so as to maximize his utility described by*

$$U\left(E\left(x_T\right), Var\left(x_T\right)\right) = E\left(x_T\right)^{\frac{1}{2}} - \exp\left(Var\left(x_T\right)\right) \tag{4.113}$$

We can have from (4.109)

$$\frac{dU}{d\gamma} = \left[\frac{1}{2\sqrt{E\left(x_T\right)}} + 2E\left(x_T\right)\exp\left(Var\left(x_T\right)\right)\right]\frac{\partial E\left(x_T\right)}{\partial\gamma} - \exp\left(Var\left(x_T\right)\right)\frac{\partial E\left(x_T^2\right)}{\partial\gamma} \tag{4.114}$$

iteration i	$\gamma^{(i)}$	$\frac{dU}{d\gamma}\|_{\gamma^{(i)}}$	$E\left(X_T\left(\gamma^{(i)}\right)\right)$	$Var\left(X_T\left(\gamma^{(i)}\right)\right)$
1	20	7.65104481	9.75964487	2.06459920
2	30	-43.75715780	14.62354562	5.06469945
3	21.4882961	7.32643305	10.48353562	2.42724372
4	22.70904576	6.57164537	11.07729784	2.74659025
5	33.33767286	-221.67735493	16.24695657	6.36072400
6	23.01506054	6.28330203	11.22614039	2.82973592
7	23.29958372	5.970601690	11.36452964	2.90815448
8	28.73245758	-21.52645896	14.00702556	4.61115178
9	24.47925856	4.09358289	11.93831177	3.24472344
10	25.15883679	2.44695775	12.26885188	3.44697565
11	26.16872008	-1.12088469	12.76004909	3.75882569
12	25.85145194	0.17300462	12.60573302	3.65939924
13	25.89387354	0.01012403	12.62636646	3.67261627
14	25.89651031	-0.00009931	12.62764896	3.67343857
15	25.89648469	-0.00000006	12.62763650	3.67343059

Table 4.13: False position method for optimal γ in Example 2

We use the false-position method again with initial values to be 0 and 20 (see Table 4.15).

t	0	1	2	3
A	4.432	4.609	4.793	4.985
B	7.190	7.477	7.776	8.087
C	25.604	26.629	27.694	28.801

Table 4.14: $v_t(\gamma)$ in Example 2

iteration	γ	$\frac{dU}{d\gamma}$	$E(X_T(\gamma))$	$Var(X_T(\gamma))$
1	0	1.394960641	0.03184338	0.03623824
2	20	-1.76509774	9.75964487	2.06459920
3	8.82870170	0.00339814	4.32603626	0.27873656
4	8.85016722	0.00267116	4.33647687	0.28058372
5	8.92903835	-0.00001707	4.67483901	0.28742319
6	8.92853753	0.00000009	4.6745941	0.28737950

Table 4.15: False position method for optimal γ in Example 3

The value of $\mathbf{K}_t = \begin{bmatrix} 0.4004 \\ 0.6496 \\ 2.3133 \end{bmatrix}$, $\forall t = 0, 1, 2, 3$ and the corresponding $v_t(\gamma), \forall t = 0, 1, 2, 3$, are given in Table 4.16.

The investor in the Example 2 is less risk averse than the one in Example 3. The investor in Example 3 places more weight on the variance-related term.

t	0	1	2	3
A	1.5820	1.5891	1.6568	1.7189
B	2.4789	2.5780	2.6811	2.7884
C	8.8278	9.1810	9.5482	9.9301

Table 4.16: $v_t(\gamma)$ in Example 3

Therefore, it is reasonable to see, from the results, that the relative proprotion of wealth at any time t invested in risky assets is much less for investor in Example 3 than in Example 2. This reflects that the investor in Example 3 is much more unwilling to make risky investment at any time than the investor in Example 2.

Chapter 5

Dynamic Portfolio Selection without Risk-less Assets

Although we usually regard government bonds are of no defualt risk and are guaranteed to have fixed income, these bonds are subject to interest rate risk and other risks. In other words, investment situations with one riskfree asset is just one of the ideal cases. And hence, we should develop models that generalize the dynamic portfolio selection problem to cover more complicated real situations, in which the risk-free asset is no longer existing.

We consider a capitial market consisted of $(n+1)$ risky securities with random returns. An investor joins the market at time period 0 with an initial wealth x_0. The investor can allocate his wealth among the $(n+1)$ assets. The wealth can be reinvested at the beginning of each of the following $(T-1)$ consecutive time periods. The rates of return of the risky securities at any time period t within the planning horizon are denoted by a vector $\mathbf{e}_t = [e_t^0, e_t^1, ..., e_t^n]'$, where e_t^i is the random return for security i at time t. It is assumed that vectors $\mathbf{e}_t$, $t = 0, 1, \ldots, T - 1$, are statistically independent and return $\mathbf{e}_t$ has mean $E(\mathbf{e}_t) = [E(e_t^0), E(e_t^1), ..., E(e_t^n)]'$ and covariance

$$Cov(\mathbf{e}_t) = \begin{bmatrix} \sigma_{t,00} & \sigma_{t,01} & \sigma_{t,02} & \cdots & \sigma_{t,0n} \\ \sigma_{t,10} & \sigma_{t,11} & \sigma_{t,12} & \cdots & \sigma_{t,1n} \\ \sigma_{t,20} & \sigma_{t,21} & \sigma_{t,22} & \cdots & \sigma_{t,2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{t,n0} & \sigma_{t,n1} & \sigma_{t,n2} & \cdots & \sigma_{t,nn} \end{bmatrix}.$$

The investor is seeking for investment strategies, $\mathbf{u}_t = [u_t^1, u_t^2, ..., u_t^n]'$ for $t = 0, 1, 2, \ldots, T-1$, such that his investment objective is optimized. Taking security 0 as reference, we have the following,

$$P_t^i = e_t^i - e_t^0 \quad i = 1, 2, ..., n \tag{5.1}$$

The wealth dynamic can be described by the following stochastic difference equation

$$\begin{aligned} x_{t+1} &= \sum_{i=1}^{n} e_t^i u_t^i + \left(x_t - \sum_{i=1}^{n} u_t^i \right) e_t^0 &\quad (5.2) \\ &= e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \qquad t = 0, 1, ..., T-1 &\quad (5.3) \end{aligned}$$

where $\mathbf{P}_t = [P_t^1, P_t^2, ..., P_t^n]' = [(e_t^1 - e_t^0), (e_t^2 - e_t^0), ..., (e_t^n - e_t^0)]'$.

It is assumed that $E(\mathbf{e}_t\mathbf{e}_t')$ is positive definite for all time periods, i.e.,

$$E(\mathbf{e}_t\mathbf{e}_t') = \begin{bmatrix} E\left((e_t^0)^2\right) & E(e_t^0e_t^1) & \cdots & E(e_t^0e_t^n) \\ E(e_t^1e_t^0) & E\left((e_t^1)^2\right) & \cdots & E(e_t^1e_t^n) \\ \vdots & \vdots & \ddots & \vdots \\ E(e_t^ne_t^0) & E(e_t^ne_t^1) & \cdots & E\left((e_t^n)^2\right) \end{bmatrix} > 0 \qquad (5.4)$$

Then, the following is true.

$$\begin{bmatrix} E\left((e_t^0)^2\right) & E(e_t^0\mathbf{P}_t') \\ E(e_t^0\mathbf{P}_t) & E(\mathbf{P}_t\mathbf{P}_t') \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 1 \end{bmatrix} E(\mathbf{e}_t\mathbf{e}_t') \begin{bmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} > 0 \qquad (5.5)$$

Further, we have the following from (5.5),

$$E(\mathbf{P}_t\mathbf{P}_t') > 0 \qquad \forall t \qquad (5.6)$$

and

$$E\left(\left(e_t^0\right)^2\right) - E\left(e_t^0\mathbf{P}_t'\right)E^{-1}(\mathbf{P}_t\mathbf{P}_t')\,E\left(e_t^0\mathbf{P}_t\right) > 0 \qquad \forall t \qquad (5.7)$$

A dynamic policy is an investment sequence,

$$\pi = (\mu_0, \mu_1, ..., \mu_{T-1}) \qquad (5.8)$$

$$= \left(\begin{bmatrix} \mu_0^1 \\ \mu_0^2 \\ \vdots \\ \mu_0^n \end{bmatrix}, \begin{bmatrix} \mu_1^1 \\ \mu_1^2 \\ \vdots \\ \mu_1^n \end{bmatrix}, \ldots, \begin{bmatrix} \mu_{T-1}^1 \\ \mu_{T-1}^2 \\ \vdots \\ \mu_{T-1}^n \end{bmatrix} \right) \tag{5.9}$$

More specifically, μ_t maps the wealth at the beginning of the t-th period, x_t, into a portfolio decision in the t-th period.

$$\begin{bmatrix} u_t^1 \\ u_t^2 \\ \vdots \\ u_t^n \end{bmatrix} = \begin{bmatrix} \mu_t^1(x_t) \\ \mu_t^2(x_t) \\ \vdots \\ \mu_t^n(x_t) \end{bmatrix} \tag{5.10}$$

A dynamic portfolio policy, π^*, is said to be efficient if there exists no other dynamic portfolio policy, π, such that $E(x_T)|_\pi \geq E(x_T)|_\pi^*$ and $Var(x_T)|_\pi \leq Var(x_T)|_\pi^*$ with at least one strict equality.

We formulate the following multi-period portfolio selection problem in order to generate efficient portfolio policies,

$$E(w) \quad : \quad \max E(x_T) - wVar(x_T) \tag{5.11}$$

$$s.t.\ x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \tag{5.12}$$

where $w > 0$. By varying the value of w in $(E(w))$, the set of efficient dynamic portfolio policies can be generated. Define $\prod_E(w)$ to be the set of optimal solutions of problem $(E(w))$ with given w, i.e.

$$\Pi_E(w) = (\pi | \pi \text{ is a maximizer of } (E(w))) \tag{5.13}$$

5.1 Construction of Auxiliuary Problem

Problem $(E(w))$ is difficult to be solved directly due to nonseparability in sense of dynamic programming. A solution scheme as the same in Chapter 4 is to embed problem $E(w)$ into a tractable auxiliary problem that is separable, investigate the relationship between the solution sets of problem $(E(w))$ and the auxiliary problem, and search the solution of the auxiliary problem that attains the optimum point of problem $(E(w))$.

Define

$$\begin{aligned}
& \tilde{U}\left(E\left(x_T^2\right), E\left(x_T\right)\right) \\
= \ & E\left(x_T\right) - wVar\left(x_T\right) \\
= \ & E\left(x_T\right) - w\left[E\left(x_T^2\right) - E^2\left(x_T\right)\right] \\
= \ & -wE\left(x_T^2\right) + \left[wE^2\left(x_T\right) + E\left(x_T\right)\right]
\end{aligned} \tag{5.14}$$

It is obvious that $\tilde{U}$ is a convex function of $E(x_T^2)$ and $E(x_T)$. We can apply theorems in chapter 4 directly. The following auxiliary problem is now constructed for $(E(w))$,

$$A(\lambda, w) \quad : \quad \max E\left(-wx_T^2 + \lambda x_T\right)$$

$$s.t.\ x_{t+1} \;=\; e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \tag{5.15}$$

Prominent features of problem $(A(\lambda, w))$ are that $(A(\lambda, w))$ is of separable structure and the objective function of $(A(\lambda, w))$ is of a quadratic form whereas the system dynamic is of a linear form. Define $\Pi_A(\lambda, w)$ to be the set of the optimal solutions of problem $(A(\lambda, w))$ for given λ and w, i.e.,

$$\Pi_A(\lambda, w) = (\pi | \pi \text{ is a maximizer of } (A(\lambda, w))) \tag{5.16}$$

5.2 Analytical Solution for Efficient Frontier

Optimal solution of the auxiliary problem $(A(\lambda, w))$ can be derived analytically by dynamic programing.

$$\max E\left(-w x_T^2 + \lambda x_T\right) \tag{5.17}$$

$$s.t.: \quad x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \tag{5.18}$$

with $w > 0$ and λ are given. As is assumed before, the statistics of first- and second-order moments of all random returns are known.

Solving problem (5.18) by dynamic programming starting form $T-1$, we have

$$\begin{aligned}
& \max\ J_{T-1}(x_{T-1}) \\
= \;& \max E\left(-w\left(e_{T-1}^0 x_{T-1} + \mathbf{P}_{T-1}' \mathbf{u}_{T-1}\right)^2 + \lambda\left(e_{T-1}^0 x_{T-1} + \mathbf{P}_{T-1}' \mathbf{u}_{T-1}\right)\right) \\
= \;& \max -w x_{T-1}^2 E\left(\left(e_{T-1}^0\right)^2\right) - 2w x_{T-1} E\left(e_{T-1}^0 \mathbf{P}_{T-1}'\right) \mathbf{u}_{T-1} \qquad (5.19) \\
& -w \mathbf{u}_{T-1}' E\left(\mathbf{P}_{T-1} \mathbf{P}_{T-1}'\right) \mathbf{u}_{T-1} + \lambda x_{T-1} E\left(e_{T-1}^0\right) + \lambda E\left(\mathbf{P}_{T-1}'\right) \mathbf{u}_{T-1}
\end{aligned}$$

Take derivative with respect to u_{T-1}, we can futher have the following.

$$
\begin{aligned}
& \frac{\partial J_{T-1}\left(x_{T-1}\right)}{\partial u_{T-1}} \\
= & -2wx_{T-1}E\left(e^0_{T-1}\mathbf{P}_{T-1}\right) - 2wE\left(\mathbf{P}'_{T-1}\mathbf{P}_{T-1}\right)\mathbf{u}_{T-1} \\
& +\lambda E\left(\mathbf{P}_{T-1}\right)
\end{aligned}
\tag{5.20}
$$

Hence,

$$
\mathbf{u}^*_{T-1} = E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}\left(\frac{\lambda}{2w}E\left(\mathbf{P}_{T-1}\right) - x_{T-1}E\left(e^0_{T-1}\mathbf{P}_{T-1}\right)\right) \tag{5.21}
$$

Subsisting (5.21) into (5.19) yields the optimal cost-to-go at $x_T - 1$,

$$
\begin{aligned}
& J^*_{T-1}\left(x_{T-1}\right) \\
= & \; x^2_{T-1}\left[-wE\left(\left(e^0_{T-1}\right)^2\right) + wE\left(e^0_{T-1}\mathbf{P}'_{T-1}\right)E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}E\left(e^0_{T-1}\mathbf{P}_{T-1}\right)\right] \\
+ & \; x_{T-1}\left[\lambda E\left(\left(e^0_{T-1}\right)\right) - \lambda E\left(\mathbf{P}'_{T-1}\right)E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}E\left(e^0_{T-1}\mathbf{P}_{T-1}\right)\right] \\
+ & \; \frac{\lambda^2}{4w}E\left(\mathbf{P}'_{T-1}\right)E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}E\left(\mathbf{P}_{T-1}\right) \\
= & \; -w_{T-1}x^2_{T-1} + \lambda_{T-1}x_{T-1} + \alpha_{T-1}E\left(\mathbf{P}'_{T-1}\right)E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}E\left(\mathbf{P}_{T-1}\right)
\end{aligned}
\tag{5.22}
$$

where

$$
\begin{aligned}
& w_{T-1} \\
= & \; w\left[E\left(\left(e^0_{T-1}\right)^2\right) - E\left(e^0_{T-1}\mathbf{P}'_{T-1}\right)E\left(\mathbf{P}_{T-1}\mathbf{P}'_{T-1}\right)^{-1}E\left(e^0_{T-1}\mathbf{P}_{T-1}\right)\right]
\end{aligned}
\tag{5.23}
$$

$$
\lambda_{T-1}
$$

$$= \lambda \left[E\left(e_{T-1}^{0}\right) - E\left(\mathbf{P}_{T-1}'\right) E\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)^{-1} E\left(e_{T-1}^{0}\mathbf{P}_{T-1}\right) \right]$$

(5.24)

$$\alpha_{T-1} = \frac{\lambda^2}{4w} \tag{5.25}$$

Since the new derived utility function has a similar form at stage t, $0 \leq t \leq T-1$, to the original utility form at stage T. We can derive the optimal control and the optimal cost-to-go at $t = T-2, ..., 1, 0$, in a similar way,

$$\mathbf{u}_t^* = E\left(\mathbf{P}_t\mathbf{P}_t'\right)^{-1} \left(\frac{\lambda_{t+1}}{2w_{t+1}} E\left(\mathbf{P}_t\right) - x_t E\left(e_t^0 \mathbf{P}_t\right) \right) \tag{5.26}$$

$$J_t^*\left(x_t\right) = -w_t x_t^2 + \lambda_t x_t + \sum_{k=t}^{T-1} \alpha_k E\left(\mathbf{P}_k'\right) E\left(\mathbf{P}_k\mathbf{P}_k'\right)^{-1} E\left(\mathbf{P}_k\right) \tag{5.27}$$

where the recursive equations for the parameters are

$$w_t = w_{t+1} \left[E\left(\left(e_t^0\right)^2\right) - E\left(e_t^0 \mathbf{P}_t'\right) E\left(\mathbf{P}_t\mathbf{P}_t'\right)^{-1} E\left(e_t^0 \mathbf{P}_t\right) \right] \tag{5.28}$$

$$\lambda_t = \lambda_{t+1} \left[E\left(\left(e_t^0\right)\right) - E\left(\mathbf{P}_t'\right) E\left(\mathbf{P}_t\mathbf{P}_t'\right)^{-1} E\left(e_t^0 \mathbf{P}_t\right) \right] \tag{5.29}$$

$$\alpha_t = \frac{\lambda_{t+1}^2}{4w_{t+1}} \tag{5.30}$$

with boundary condition given as

$$w_T = w \tag{5.31}$$

$$\lambda_T = \lambda \tag{5.32}$$

The optimal portfolio policy for auxiliary problem $(A(\lambda, w))$ at each time period t can be written as,

$$\mathbf{u}_t^*\left(x_t; \gamma\right) = -\mathbf{K}_t x_t + \mathbf{v}_t\left(\gamma\right) \qquad t = 0, 1, \ldots, T-1 \tag{5.33}$$

where

$$\gamma = \frac{\lambda}{w} \tag{5.34}$$

$$\mathbf{K}_t = E^{-1}\left(\mathbf{P}_t'\mathbf{P}_t\right) E\left(e_t^0\mathbf{P}_t\right) \tag{5.35}$$

$$\mathbf{v}_t(\gamma) = \frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{E\left(e_k^0\right)-\widehat{B}_k}{E\left(\left(e_k^0\right)^2\right)-\widetilde{B}_k}\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(\mathbf{P}_t\right)$$

$$t = 0, 1, 2, .., T-2 \tag{5.36}$$

with boundary condition,

$$\mathbf{v}_{T-1}(\gamma) = \frac{\gamma}{2}E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)^{-1}E\left(\mathbf{P}_{T-1}\right) \tag{5.37}$$

and

$$B_t = E\left(\mathbf{P}_t'\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(\mathbf{P}_t\right) \tag{5.38}$$

$$\widehat{B}_t = E\left(\mathbf{P}_t'\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(e_t^0\mathbf{P}_t\right) \tag{5.39}$$

$$\widetilde{B}_t = E\left(e_t^0\mathbf{P}_t'\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(e_t^0\mathbf{P}_t\right) \tag{5.40}$$

Substituting (5.33) into the wealth dynamics (5.3) yields the dynamics of the wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$x_{t+1}(\gamma) = \left(e_t^0 - \mathbf{P}_t'\mathbf{K}_t\right)x_t(\gamma) + \mathbf{P}_t'\mathbf{v}_t(\gamma) \tag{5.41}$$

Taking expections on both sides of (5.41) and noticing the statistical independence between $(e_t^0, \mathbf{P}_t)$ and x_t, we have the following recursive expression for the expected wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$E\left(x_{t+1}(\gamma)\right) = \left(E\left(e_t^0\right) - \widehat{B}_t\right)E\left(x_t(\gamma)\right)$$

$$+\frac{\gamma}{2}\left(\prod_{k=t+1}^{T-1}\frac{E\left(e_k^0\right)-\widehat{B}_k}{E\left(\left(e_k^0\right)^2\right)-\widetilde{B}_k}\right)B_t \tag{5.42}$$

Taking square on both sides of (5.41) yields,

$$\begin{aligned}x_{t+1}^2(\gamma) &= \left[\left(e_t^0\right)^2-2e_t^0\mathbf{P}_t'\mathbf{K}_t+\mathbf{K}_t'\mathbf{P}_t\mathbf{P}_t'\mathbf{K}_t\right]x_t^2(\gamma)\\ &\quad +2\left(e_t^0-\mathbf{P}_t'\mathbf{K}_t\right)x_t(\gamma)\,\mathbf{P}_t'\mathbf{v}_t(\gamma)+\mathbf{v}_t(\gamma)'\,\mathbf{P}_t\mathbf{P}_t'\mathbf{v}_t(\gamma)\\ &\qquad t=0,1,\ldots,T-1\end{aligned} \tag{5.43}$$

Taking expection on both sides of the above equation and performing a simplification lead to the following recursive expression for the expected value of the square wealth between successive time periods under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$\begin{aligned}E\left(x_{t+1}^2(\gamma)\right) &= \left(E\left(\left(e_t^0\right)^2\right)-\widetilde{B}_t\right)E\left(x_t^2(\gamma)\right)\\ &\quad +\frac{\gamma^2}{4}\left(\prod_{k=t+1}^{T-1}\frac{E\left(e_k^0\right)-\widehat{B}_k}{E\left(\left(e_k^0\right)^2\right)-\widetilde{B}_k}\right)^2 B_t\end{aligned} \tag{5.44}$$

Solving the above recurresive equations (5.42) and (5.44), we obtain an explicit expression for the expected values of the final wealth and the square of the final wealth under under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$,

$$E\left(x_T(\gamma)\right)=\alpha_1x_0+\beta\gamma \tag{5.45}$$

$$E\left(x_T^2(\gamma)\right)=\alpha_2x_0^2+\frac{\beta}{2}\gamma^2 \tag{5.46}$$

where

$$\alpha_1 = \prod_{t=0}^{T-1}\left(E\left(e_t^0\right)-\widehat{B}_t\right) \tag{5.47}$$

$$\beta = \frac{1}{2}\sum_{t=0}^{T-1}\left[\left(\prod_{k=t+1}^{T-1}\frac{\left(E\left(e_k^0\right)-\widehat{B}_k\right)^2}{E\left(\left(e_k^0\right)^2\right)-\widetilde{B}_k}\right)B_t\right] \tag{5.48}$$

$$\alpha_2 = \prod_{t=0}^{T-1}\left(E\left(\left(e_t^0\right)^2\right)-\widetilde{B}_t\right) \tag{5.49}$$

The variance of the final wealth under portfolio policy $\mathbf{u}_t^*(x_t;\gamma)$ can be expressed in terms of γ using (5.45) and (5.46),

$$\begin{aligned} Var\left(x_T(\gamma)\right) &= E\left(x_T^2(\gamma)\right)-E^2\left(x_T(\gamma)\right) \\ &= \left(\frac{1}{2}-\beta\right)\beta\gamma^2-2\alpha_1x_0\beta\gamma+\left(\alpha_2-\alpha_1^2\right)x_0^2 \end{aligned} \tag{5.50}$$

It can be verified that $E\left(x_T(\gamma)\right)$ is an increasing linear function of γ whereas the variance of the final wealth, $Var\left(x_T(\gamma)\right)$, is a quadratic function of γ. From (5.45) and (5.46), we can express $\widetilde{U}\left(E\left(x_T^2\right),E\left(x_T\right)\right)$ as a function of γ,

$$\begin{aligned} &\widetilde{U}\left(E\left(x_T^2\right),E\left(x_T\right)\right) \\ =\ & -wE\left(x_T^2\right)+\left[wE^2\left(x_T\right)+E\left(x_T\right)\right] \\ =\ & -w\left(\frac{1}{2}-\beta\right)\beta\gamma^2+\left(2w\alpha_1x_0+1\right)\beta\gamma \\ & +\left(-w\alpha_2x_0^2+w\alpha_1^2x_0^2+\alpha_1x_0\right) \end{aligned} \tag{5.51}$$

Differentiating (5.51) with respect to γ yields,

$$\frac{d\widetilde{U}}{d\gamma} = -2w\left(\frac{1}{2}-\beta\right)\beta\gamma+\left(2w\alpha_1x_0+1\right)\beta \tag{5.52}$$

The optimal γ must satifies the optimality condition of $\frac{d\widetilde{U}}{d\gamma}=0$, i.e.,

$$\gamma^* = \frac{2w\alpha_1x_0+1}{2w\left(\frac{1}{2}-\beta\right)} \tag{5.53}$$

The solution scheme for $(E(w))$ becomes a very simple one now. Given a problem $(E(w))$, we can first calculate optimal γ using (5.53). Then we substitute the optimal γ into (5.33) to obtain the optimal dynamic portfolio policy for $(E(w))$,

$$\begin{aligned} \mathbf{u}_t^* &= -E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(e_t^0\mathbf{P}_t\right)x_t \\ &+\frac{2w\alpha_1 x_0+1}{4w\left(\frac{1}{2}-\beta\right)}\left(\prod_{k=t+1}^{T-1}\frac{E\left(e_k^0\right)-\widehat{B}_k}{E\left(\left(e_k^0\right)^2\right)-\widehat{\widetilde{B}}_k}\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(\mathbf{P}_t\right) \end{aligned} \quad (5.54)$$

The optimal portfolio policy at time period T - 1 is,

$$\begin{aligned} \mathbf{u}_{T-1}^* &= -E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(e_{T-1}^0\mathbf{P}_{T-1}\right)x_{T-1} \\ &+\frac{2w\alpha_1 x_0+1}{4w\left(\frac{1}{2}-\beta\right)}E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(\mathbf{P}_{T-1}\right) \end{aligned} \quad (5.55)$$

On the efficient frontier, the expected value and the variance of the final wealth can be expressed by

$$E\left(x_T(w)\right) = \alpha_1 x_0 + \beta\frac{2w\alpha_1 x_0+1}{2w\left(\frac{1}{2}-\beta\right)} \quad (5.56)$$

$$\begin{aligned} Var\left(x_T(w)\right) &= E\left(x_T^2(w)\right) - E^2\left(x_T^2(w)\right) \\ &= \alpha_2 x_0^2 + \frac{\beta}{2}\frac{\left(2w\alpha_1 x_0+1\right)^2}{4w^2\left(\frac{1}{2}-\beta\right)^2} - [\alpha_1 x_0 + \beta\frac{2w\alpha_1 x_0+1}{2w\left(\frac{1}{2}-\beta\right)}]^2 \\ &= \left(\frac{1}{2}-\beta\right)\beta\left(\frac{2w\alpha_1 x_0+1}{2w\left(\frac{1}{2}-\beta\right)}\right)^2 \\ &-2\alpha_1 x_0\beta\frac{2w\alpha_1 x_0+1}{2w\left(\frac{1}{2}-\beta\right)} + \left(\alpha_2-\alpha_1^2\right)x_0^2 \\ &= \frac{2w^2x_0^2\left(\alpha_2-\alpha_1^2\right)+\beta\left(1-4w^2x_0^2\alpha_2\right)}{4w^2\left(\frac{1}{2}-\beta\right)} \end{aligned} \quad (5.57)$$

The efficient frontier can be obtained by eliminating the parameter w in (5.56) and (5.57),

$$
\begin{aligned}
Var\left(x_T\right) &= \left(\frac{1}{2\beta}-1\right)E^2\left(x_T\right) \\
&\quad -\frac{\alpha_1 x_0}{\beta}E\left(x_T\right)+x_0^2\left(\alpha_2+\frac{\alpha_1^2}{2\beta}\right)
\end{aligned}
\tag{5.58}
$$

Example 4 *In this example, we apply the proposed solution method to the case study in section 9.2.1 of Sharpe et. al [20]. An investor has one unit wealth in the very beginning of the planning horizon. The investor is trying to find the best allocation of his wealth among three risky securities, A, B, and C, in the following 4 periods. The statistics of the random returns are stationary. Let $\mathbf{e}=[e^A,e^B,e^C]'$. The expected returns of the risky securities, A, B and C, are $E\left(e^A\right)=1.162$, $E\left(e^B\right)=1.246$ and $E\left(e^C\right)=1.228$, respectively. The covariance is* $Cov\left(\mathbf{e}\right)=\begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}$.

When the process is stationary, i.e., the statistics of the return rates of the risky assets do not change over time, we have the following simplified formulas for all time t,

$$\widehat{B}_t = \widehat{B}=E\left(\mathbf{P}'\right)E\left(\mathbf{P}\mathbf{P}'\right)^{-1}E\left(e^0\mathbf{P}\right) \tag{5.59}$$

$$B_t = B=E\left(\mathbf{P}'\right)E\left(\mathbf{P}\mathbf{P}'\right)^{-1}E\left(\mathbf{P}\right) \tag{5.60}$$

$$\widetilde{B}_t = \widetilde{B}=E\left(e^0\mathbf{P}'\right)E\left(\mathbf{P}\mathbf{P}'\right)^{-1}E\left(e^0\mathbf{P}\right) \tag{5.61}$$

$$\alpha_1 = \left(E\left(e^0\right) - \widehat{B}\right)^T \tag{5.62}$$

$$\alpha_2 = \left(E\left(\left(e^0\right)^2\right) - \widetilde{B}\right)^T \tag{5.63}$$

$$\beta = \frac{B}{2}\sum_{t=0}^{T-1}\left(\frac{\left(E\left(e^0\right) - \widehat{B}\right)^2}{E\left(\left(e^0\right)^2\right) - \widetilde{B}}\right)^T = \frac{B}{2}\left(\frac{1-\left(\frac{\alpha_1^2}{\alpha_2}\right)^T}{1-\left(\frac{\alpha_1^2}{\alpha_2}\right)}\right) \tag{5.64}$$

By applying the formulas we derived in this section, we have

$$E(\mathbf{P}_t) = E[e_B - e_A, e_C - e_A]' = [0.084, 0.066]' \tag{5.65}$$

$$E\left(e_A^2\right) = Var\left(e_A\right) + E^2\left(e_A\right) = 1.3648 \tag{5.66}$$

$$E\left(e_B^2\right) = Var\left(e_B\right) + E^2\left(e_B\right) = 1.6379 \tag{5.67}$$

$$E\left(e_C^2\right) = Var\left(e_C\right) + E^2\left(e_C\right) = 1.5369 \tag{5.68}$$

$$E\left(e_A e_B\right) = cov\left(e_A, e_B\right) + E\left(e_A\right)E\left(e_B\right) = 1.4666 \tag{5.69}$$

$$E\left(e_A e_C\right) = cov\left(e_A, e_C\right) + E\left(e_A\right)E\left(e_C\right) = 1.4414 \tag{5.70}$$

$$E\left(e_B e_C\right) = cov\left(e_B, e_C\right) + E\left(e_B\right)E\left(e_C\right) = 1.5404 \tag{5.71}$$

$$E(\mathbf{P}_t\mathbf{P}_t') = E\begin{bmatrix} e_B^2 - 2e_Ae_B + e_A^2 & e_Be_C - e_Ae_C - e_Ae_B + e_A^2 \\ e_Be_C - e_Ae_C - e_Ae_B + e_A^2 & e_C^2 - 2e_Ae_C + e_A^2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0697 & -0.0027 \\ -0.0027 & 0.0189 \end{bmatrix} \tag{5.72}$$

$$E(\mathbf{P}_t\mathbf{P}_t')^{-1} = \begin{bmatrix} 14.4270 & 2.0610 \\ 2.0610 & 53.2045 \end{bmatrix} \tag{5.73}$$

$$E(e_A\mathbf{P}_t') = \left[E\left(e_Ae_B\right) - E\left(e_A^2\right), E\left(e_Ae_C\right) - E\left(e_A^2\right)\right] \tag{5.74}$$

$$= [0.1017, 0.0766] \tag{5.75}$$

$$B = E(\mathbf{P}')\, E(\mathbf{PP}')^{-1}\, E(\mathbf{P}) = 0.3566 \tag{5.76}$$

$$\widehat{B} = E(\mathbf{P}')\, E(\mathbf{PP}')^{-1}\, E(e_A\mathbf{P}) = 0.4196 \tag{5.77}$$

$$\widetilde{B} = E(e_A\mathbf{P}')\, E(\mathbf{PP}')^{-1}\, E(e_A\mathbf{P}) = 0.4938 \tag{5.78}$$

We can further obtain

$$\alpha_1 = \left(E\left(e^A\right) - \widehat{B}\right)^T = 0.3038 \tag{5.79}$$

$$\alpha_2 = \left(E\left(\left(e^A\right)^2\right) - \widetilde{B}\right)^T = 0.5757 \tag{5.80}$$

$$\beta = \frac{B}{2}\left(\frac{1-\left(\frac{\alpha_1^2}{\alpha_2}\right)^T}{1-\left(\frac{\alpha_1^2}{\alpha_2}\right)}\right) = 0.2122 \tag{5.81}$$

The optimal γ^* in (5.53) is

$$\begin{aligned}\gamma^* &= \frac{2w\alpha_1 x_0 + 1}{2w\left(\frac{1}{2}-\beta\right)} \\ &= 1.056 + \frac{1.737}{w}\end{aligned} \tag{5.82}$$

On the efficient frontier, the expected final return and its corresponding variance can be expressed as functions of w,

$$\begin{aligned}E\left(x_4(w)\right) &= \alpha_1 x_0 + \beta\left(\frac{2w\alpha_1 x_0 + 1}{2w\left(\frac{1}{2}-\beta\right)}\right) \\ &= 0.5279 + \frac{0.3686}{w}\end{aligned} \tag{5.83}$$

and

$$\begin{aligned}Var\left(x_4(w)\right) &= \frac{2w^2x_0^2\left(\alpha_2-\alpha_1^2\right) + \beta\left(1-4w^2x_0^2\alpha_2\right)}{4w^2\left(\frac{1}{2}-\beta\right)} \\ &= 0.4154 + \frac{0.1843}{w^2}\end{aligned} \tag{5.84}$$

The efficient frontier in this example problem is given by

$$Var\left(x_4\right) = 1.356E^2\left(x_4\right) - 1.432E\left(x_4\right) + 0.7934 \tag{5.85}$$

For a given w specified by an investor, the optimal γ is calculated using (5.45). Optimal investment policy can be then obtained from (5.33). The first term of the optimal investment policy is independent of γ while the second term is related to value of γ. In this stationary example, the two-dimensional vector $\mathbf{K}_t$ is constant and $\mathbf{K}_t = \begin{bmatrix} 1.6238 \\ 4.2907 \end{bmatrix}, t = 0, 1, 2, 3.$

When $w = 2$, i.e., the investor is to maximize an objective of $E\left(x_4\right)$ - $2Var\left(x_4\right)$. The optimal γ^* is equal to 1.9245 and $\mathbf{v}_t\left(\gamma\right)$, $t = 0,\ 1,\ 2,\ 3$, are given by $\mathbf{v}_0\left(\gamma\right) = \begin{bmatrix} 0.8023 \\ 2.1984 \end{bmatrix}$, $\mathbf{v}_1\left(\gamma\right) = \begin{bmatrix} 0.9413 \\ 2.5793 \end{bmatrix}$, $\mathbf{v}_2\left(\gamma\right) = \begin{bmatrix} 1.1044 \\ 3.0262 \end{bmatrix}$, $\mathbf{v}_3\left(\gamma\right) = \begin{bmatrix} 1.2958 \\ 3.5506 \end{bmatrix}$. The investment strategy $\mathbf{u}_t = -\mathbf{K}_t\left(x_t\right) + \mathbf{v}_t\left(\gamma\right)$ is depending on the realization of wealth at each period, x_t. The investment in the first security, asset A, at period t is equal to $(x_t - \sum u_t^i)$. The corresponding $E(x_4)$ and $Var(x_4)$ are 0.7122 and 0.4615 respectively.

When $w = 10$, i.e., the investor is to maximize an objective of $E\left(x_4\right) - 10Var\left(x_4\right)$. The optimal γ^* is equal to 1.2295 and $\mathbf{v}_t\left(\gamma\right)$, $t = 0,\ 1,\ 2,\ 3$, are given by $\mathbf{v}_0\left(\gamma\right) = \begin{bmatrix} 0.5126 \\ 1.4045 \end{bmatrix}$, $\mathbf{v}_1\left(\gamma\right) = \begin{bmatrix} 0.6014 \\ 1.6478 \end{bmatrix}$, $\mathbf{v}_2\left(\gamma\right) = \begin{bmatrix} 0.7056 \\ 1.9334 \end{bmatrix}$, $\mathbf{v}_3\left(\gamma\right) =$

$\begin{bmatrix} 0.8278 \\ \\ 2.2684 \end{bmatrix}$. While the corresponding $E\left(x_4\right)$ is 0.5648 and $Var\left(x_4\right)$ is 0.4172.

Since return of security A is having the smallest variance among the three securities under consideration, we can regard it as the one of least risk. The investment in security A, u_t^A, can be expressed in the following equation.

$$\begin{aligned} u_t^A &= x_t - \left(-\left(\sum K_t^i\right)x_t + \sum v_t^i\left(\gamma\right)\right) && (5.86) \\ &= x_t\left(1 + \sum K_t^i\right)x_t - \sum v_t^i\left(\gamma\right) && (5.87) \\ &= -K_t^A x_t + v_t^A\left(\gamma\right) && (5.88) \end{aligned}$$

Similar to situation in where we have a risk-free asset, the proportion in investment of least risky asset (security A)i is $\frac{-K_t^i x_t + v_t^i(\gamma)}{x_t}$. Since $v_t^i\left(\gamma\right)$ is independent of x_t and K_t is a constant vector over time, a larger value of $v_t^i\left(\gamma\right)$ will induce a relatively larger proportion of wealth at period t invested in security i.

We have observed from the results that the value of $v_t^i\left(\gamma\right)$ in any risky asset i decreases as the vaule of w increases and, on the other hand, the value of $v_t^A\left(\gamma\right)$ decreases. That confirms that the investor is more unwilling to invest in relatively risky assets when he/she is more risk-averse.

As we can see that the values of $v_t^i\left(\gamma\right)$, $\forall t$, are increasing over time, we can find that the investor increases the relative proportion of his investment in all the relatively risky assts when the time is getting closer to the planning horizon . We guess that when the time is approaching to the end of planning horizon T, the future becomes clearer and the investor becomes less cautious. This result is as the same as the "time effect" in Mossin[6] where a maximization of quadratic utility of terminal wealth is studied.

Finally, we can see that all future information are used in the optimal investment policy. We can conclude that the optimal policy is completely non-myopic.

5.3 Reduction to Investment Situations with One Risk-free Asset

Investment situations where there exists a risk-less asset can be regarded as a special case in this general framework of multi-period mean-variance analysis. Let the zeroth security be risk-less. Then we have $e_t^0 = s_t$ and $cov\left(e_t^0, e_t^i\right) = 0, \forall i = 0, 1, ..., n$. We further have the following

$$
\begin{aligned}
E\left(e_t^0 \mathbf{P}_t\right) &= s_t E\left(\mathbf{P}_t\right) && (5.89)\\
\widehat{B}_t &= s_t B_t && (5.90)\\
\widetilde{B}_t &= s_t^2 B_t && (5.91)\\
\alpha_1 &= \prod_{t=0}^{T-1}\left(E\left(e_t^0\right) - \widehat{B}_t\right) \\
&= \prod_{t=0}^{T-1} s_t\left(1 - B_t\right) && (5.92)\\
\alpha_2 &= \prod_{t=0}^{T-1}\left(E\left(\left(e_t^0\right)^2\right) - \widetilde{B}_t\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{t=0}^{T-1} s_t^2 (1 - B_t) && (5.93)\\
\beta &= \frac{1}{2}\sum_{t=0}^{T-1}\left[\left(\prod_{k=t+1}^{T-1}\frac{\left(E\left(e_k^0\right)-\widehat{B}_k\right)^2}{E\left(\left(e_k^0\right)^2\right)-\widetilde{B}_k}\right)B_t\right]\\
&= \frac{1}{2}\sum_{t=0}^{T-1}\left(\prod_{k=t+1}^{T-1}(1-B_k)\,B_t\right)\\
&= \frac{1}{2}\left(1-\prod_{t=0}^{T-1}(1-B_t)\right) && (5.94)
\end{aligned}
$$

The optimal parameter γ in the investment situations with a risk-less asset becomes

$$
\begin{aligned}
\gamma^* &= \frac{2w\prod_{t=0}^{T-1} s_t (1-B_t)\,x_0 + 1}{w\left(\prod_{t=0}^{T-1}(1-B_t)\right)}\\
&= 2\prod_{t=0}^{T-1} s_t x_0 + \frac{1}{w\left(\prod_{t=0}^{T-1}(1-B_t)\right)} && (5.95)
\end{aligned}
$$

The optimal portfolio policies for $t = 0, 1, 2, .., T-2$ are

$$
\begin{aligned}
&\mathbf{u}_t^* = -s_t E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(\mathbf{P}_t\right)x_t\\
&+\frac{2w\prod_{t=0}^{T-1} s_t (1-B_t)\,x_0 + 1}{2w\left(\prod_{t=0}^{T-1}(1-B_t)\right)}\left(\prod_{k=t+1}^{T-1}\frac{1}{s_k}\right)E^{-1}\left(\mathbf{P}_t\mathbf{P}_t'\right)E\left(\mathbf{P}_t\right) \qquad (5.96)
\end{aligned}
$$

The optimal portfolio policy at time period T - 1 is,

$$
\begin{aligned}
&\mathbf{u}_{T-1}^* = -s_{T-1}E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(\mathbf{P}_{T-1}\right)x_{T-1}\\
&+\frac{2w\prod_{t=0}^{T-1} s_t (1-B_t)\,x_0 + 1}{2w\left(\prod_{t=0}^{T-1}(1-B_t)\right)}E^{-1}\left(\mathbf{P}_{T-1}\mathbf{P}_{T-1}'\right)E\left(\mathbf{P}_{T-1}\right) \qquad (5.97)
\end{aligned}
$$

The expected final wealth under the optimal portfolio policy is given by

$$
\begin{aligned}
E\left(x_T\right) &= \alpha_1 x_0 + \beta\gamma^*\\
&= \prod_{t=0}^{T-1} s_t (1-B_t)\,x_0
\end{aligned}
$$

$$
\begin{aligned}
& + \frac{2w \prod_{t=0}^{T-1} s_t (1 - B_t) x_0 + 1}{2w \left(\prod_{t=0}^{T-1} (1 - B_t)\right)} \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right) \\
= & \prod_{t=0}^{T-1} s_t x_0 + \frac{\left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right)}{2w \left(\prod_{t=0}^{T-1} (1 - B_t)\right)} \qquad (5.98)
\end{aligned}
$$

The variance of the final wealth under the optimal portfolio policy is given by

$$
\begin{aligned}
Var(x_T) = & \left(\frac{1}{2} - \beta\right) \beta (\gamma^*)^2 - 2\alpha_1 x_0 \beta \gamma^* + \left(\alpha_2 - \alpha_1^2\right) x_0^2 \\
& = \frac{1}{2} \left(\prod_{t=0}^{T-1} (1 - B_t)\right) \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right) (\gamma^*)^2 \\
& - \left(\prod_{t=0}^{T-1} s_t (1 - B_t)\right) \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right) x_0 \gamma^* \\
& + \left(\prod_{t=0}^{T-1} s_t^2 (1 - B_t) - \prod_{t=0}^{T-1} s_t^2 (1 - B_t)^2\right) x_0^2 \\
& = \prod_{t=0}^{T-1} (1 - B_t) \left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right) \left(\frac{\gamma}{4} - x_0 \prod_{t=0}^{T-1} s_t\right)^2 \qquad (5.99) \\
& = \frac{\left(1 - \prod_{t=0}^{T-1} (1 - B_t)\right)}{4w^2 \prod_{t=0}^{T-1} (1 - B_t)} \qquad (5.100)
\end{aligned}
$$

Finally, the analytical expression of the efficient frontier in (5.58) can be reduced to the following simpler form for situation with a risk-less asset,

$$
Var(x_T) = \frac{\prod_{t=0}^{T-1} (1 - B_t)}{1 - \prod_{t=0}^{T-1} (1 - B_t)} \left(E(x_T) - x_0 \prod_{t=0}^{T-1} s_t\right)^2 \qquad (5.101)
$$

5.4 Multi-period Portfolio Selection via Maximizing Utility function $U(E(x_T), Var(x_T))$

We consider in this section a more general problem formulation for multi-period portfolio selection. The objective of an investor is to maximize his utility function that is dependent on the expected value and the variance of the terminal wealth x_T, $U\left(E\left(x_T\right), Var\left(x_T\right)\right)$. Since investors always would like to maximize their final wealth with low risk level, utility function $U\left(E\left(x_T\right), Var\left(x_T\right)\right)$ is assumed to satisfy the following,

$$\frac{\partial U\left(E\left(x_T\right), Var\left(x_T\right)\right)}{\partial E\left(x_T\right)} > 0 \tag{5.102}$$

and

$$\frac{\partial U\left(E\left(x_T\right), Var\left(x_T\right)\right)}{\partial Var\left(x_T\right)} < 0 \tag{5.103}$$

The following mutli-period portfolio selection problem is formulated

$$(U) : \max \qquad U\left(E\left(x_T\right), Var\left(x_T\right)\right) \tag{5.104}$$

$$\text{s.t.} : x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \tag{5.105}$$

Define Π_U to be the set of the optimal solution of problem (U), i.e.

$$\Pi_U = (\pi | \pi \text{ is the maximizer of } (U)) \tag{5.106}$$

Problem formulation (U) covers a general class of multi-period portfolio selection problem. An utility function, in gerneral, can be nonlinear with respect to $E\left(x_T\right)$ and $Var\left(x_T\right)$. The multi-period mean variance formulation discussed in the previous sections can be seen as a special case of problem formulation (U) where the utiliyt function is linear with respect to $E\left(x_T\right)$ and $Var\left(x_T\right)$.

Theorems in chapter 4 concerning general class of utility functions can be applied directly. The optimal solution for problem $(A\left(\lambda, w\right))$ was derived for given

$\gamma = \frac{\lambda}{w}$ in the previous sections. The computional procedure to obtain the optimal γ is now constructed by studying the derivative of U with respect to γ. The derivative of the utility function with respect to γ can be obtained using the following formula,

$$\frac{dU}{d\gamma} = \left(\frac{\partial U}{\partial E\left(x_T\right)} - 2E\left(x_T\right)\frac{\partial U}{\partial Var\left(x_T\right)}\right)\frac{dE\left(x_T\right)}{d\gamma} + \frac{\partial U}{\partial Var\left(x_T\right)}\frac{dE\left(x_T^2\right)}{d\gamma} \tag{5.107}$$

where $\frac{dE(x_4)}{d\gamma} = \beta$ and $\frac{dE\left(x_4^2\right)}{d\gamma} = \beta\gamma$ as seen from (5.45) and (5.46) respectively.

By setting $\frac{dU}{d\gamma}$ in (5.107) equal to zero, we have the following necessary optimum condition for γ,

$$\left(\frac{\partial U}{\partial E\left(x_T\right)} - 2E\left(x_T\right)\frac{\partial U}{\partial Var\left(x_T\right)}\right) + \frac{\partial U}{\partial Var\left(x_T\right)}\gamma = 0 \tag{5.108}$$

i.e.

$$\gamma = 2E\left(x_T\right) - \frac{\partial U}{\partial E\left(x_T\right)} / \frac{\partial U}{\partial Var\left(x_T\right)} \tag{5.109}$$

Since both $E\left(x_T\right)$ and $Var\left(x_T\right)$ are dependent on parameter γ, (5.109) is of little use in finding the optimal γ in the calculation process. It can be used for checking the optimality. As the derivative of $\frac{dU}{d\gamma}$ is obtainable for given γ, a numerial search method using gradient information, such as the gradient method or the false position method, can be adopted to find the optmal value of γ. We summarize the general algorithm for dynamic mean-variance analysis below :

Alogrithm

1. Set iteration number $j = 0$, choose an initial value $\gamma^{(0)}$ and select a very small value for ε

2. For given $\gamma^{(j)}$, use (5.45) and (5.50) to obtain $E\left(x_T\left(\gamma^{(j)}\right)\right)$ and $Var\left(x_T\left(\gamma^{(j)}\right)\right)$ respectively. Calculate $\frac{dU}{d\gamma}|_{\gamma=\gamma^{(j)}}$ using (5.107).

3. If $\left|\frac{dU}{d\gamma}|_{\gamma=\gamma^{(j)}}\right| < \varepsilon$, set $\gamma^* = \gamma^{(j)}$, goto step 4.

   else, update the value of γ to $\gamma^{(j+1)}$ using gradient method, false position method or any other suitable method. Set $j = j + 1$, goto step 2.

4. Calculate the optimal control portfolio policies at various periods using (5.33)-(5.40) with $\gamma^* = \gamma^{(j)}$

Example 5 *Consider Example 4 again. But this time the investor seeks an optimal portfolio policy that maxmizes the following utility function*

$$U\left(E\left(x_4\right), Var\left(x_4\right)\right) = E^2\left(x_4\right) - \exp\left[Var\left(x_4\right)\right]$$

The derivative of U with repsect to γ can be obtained from (5.107).

$$\begin{aligned} \frac{dU}{d\gamma} &= 2E\left(x_4\right)\left[1 + \exp\left(Var\left(x_4\right)\right)\right]\frac{dE\left(x_4\right)}{d\gamma} \\ &\quad - \exp\left(Var\left(x_4\right)\right)\frac{dE\left(x_4^2\right)}{d\gamma} \end{aligned} \tag{5.110}$$

The initial value of γ is set to 2. The optimal value of γ^* is found to be 2.7649 using a gradient search with step-size 0.1 and ε to be 0.0001 . The two-dimensional vector of $\mathbf{K}_t$ is $\begin{bmatrix} 1.6238 \\ 4.2907 \end{bmatrix}$ for $t = 0, 1, 2, 3$. The second term in the optimal portfolio policy, $\mathbf{v}_t(\gamma)$, $t = 0, 1, 2, 3$, are given by $\mathbf{v}_0(\gamma) = \begin{bmatrix} 1.1527 \\ 3.1584 \end{bmatrix}$,

$\mathbf{v}_1(\gamma) = \begin{bmatrix} 1.3524 \\ 3.7057 \end{bmatrix}$, $\mathbf{v}_2(\gamma) = \begin{bmatrix} 1.5867 \\ 4.3478 \end{bmatrix}$ $\mathbf{v}_t(\gamma) = \begin{bmatrix} 1.8617 \\ 5.1011 \end{bmatrix}$.

The idea presented in this section has been successful applied to Multi-period portfolio selection problem in which the investor is using a "safety-first" approach[37]. The safety-first approach is firstly proposed by Roy[38]. An safety-first investor is trying to avoid him/her from trapping into a "disaster". In other words, he/she is constructing portfolio so as to minimize the probabilty $P(x_T) \leq d$ where d is called as a "disaster" level from his/her point of view.

The safety-first approach is closely related to Mean-Variance approach[39]. From Bienaymé-Tchebycheff inequlity, we have $P(x_T \leq d) \leq \frac{Var(x_T)}{(E(x_T)-d)^2}$. Thus, minimizing $P(x_T \leq d)$ can be achieved by maximizing $\frac{E(x_T)-d}{\sqrt{Var(x_T)}}$ as suggested in Roy[38]. A multi-period safety-first portfolio selection problem can then be formulated as :

$$\begin{aligned} (P) \qquad & maxU\left(E\left(x_T\right), Var\left(x_T\right)\right) = \frac{\left(E\left(x_T\right) - d\right)}{\sqrt{Var\left(x_T\right)}} \\ & \text{s.t.} : x_{t+1} = e_t^0 x_t + \mathbf{P}_t' \mathbf{u}_t \quad t = 0, 1, 2, ..., T-1 \end{aligned} \tag{5.111}$$

It is clear that U satisfies $\frac{\partial U}{\partial E(x_t)} > 0$ for all $E(x_T)$ and $Var(x_T)$ and $\frac{\partial U}{\partial Var(x_T)} < 0$ for all $Var(x_T)$ and $E(x_T) > d$. Analytic expression of optimal investment policies are achieved by applying the results discussed (see [37]).

Chapter 6

Conclusions and Recommendations

6.1 Summaries and Achievements

Mean-variance approach has been widely adopted in single-period portfolio selection, because the mean-variance model is intuitive for investors to make investment decisions. The mean-variance analysis uses the mean as the measurement of return and variance as the proxy for risk. Any investor is therefore able to make a trade-off between the expected return and the risk quantitatively. If all investors are risk-aversion while having the same information, there exists only one efficient frontier. For investors of different degrees of risk aversion, the mean-variance approach simply matches the utility function for a particular investor to the unique efficient frontier such that the best-compromised portfolio can be found for the particular investor.

There is a gap in applying the mean-variance approach in multi-period portfolio selection. The previous literuatures in multi-period analysis require the assumption of separable utility function. Therefore, the philosophy of the mean-variance approach is seldomly applied in dynamic portfolio management as variance-minimization is a notorious problem in dynamic programming.

The objective of this research is to seek an implementable framework that can further extend the spirit of mean-variance model to dynamic stochastic investment environment. We have introduced an iterative solution scheme that is capable of solving non-separable problem by parametric dynamic programming. The efficient searching scheme is proposed by making use of the derivative of utility function with respect to the introduced parameter. Analytical expression for multi-period mean-variance efficienct frontier is successfully achieved. The dynamic control policy for multi-period model can also be obtained explicitly after searching for optimal parameter. Futhermore, the proposed framework is able to solve a general class of multi-period portfolio selection problems with a nonlinear utility function of the mean and the variance of the terminal wealth.

6.2 Future Studies

6.2.1 Constrained Investment Situations

In the mathematical model that is adopted for multi-period portfolio selection problems studied in this thesis, it is assumed that the decision variables are unbounded. However, there exist constraints on investment in most real-world situations.

Similar to the unconstrained dynamic investment, the primal objective of an investor with a given degree of risk aversion is to maximize a utility of the mean and the variance of the final wealth, $U\left(E\left(x_T\right), Var\left(x_T\right)\right)$. Applying the same solution concept and the same parametric solution scheme, we can prove that the optimal portfolio policy can be sought by the auxiliary problem $\left(A\left(\lambda, w\right)\right)$. For the constrained investment situations, the auxiliary parametric model now becomes

$$\max \quad E\left(-wx_T^2 + \lambda x_T\right) \tag{6.1}$$

$$s.t. \quad x_{t+1} = s_t x_t + P_t' u_t \quad t = 0, 1, ..., T-1 \tag{6.2}$$

$$a_t \leq u_t \leq b_t \tag{6.3}$$

The above formulation is a stochastic quadratic programming problem. Different sets of active assets are present corresponding to different current wealth. The optimal cost-to-go becomes a piecewise quadratic function of the current wealth. The number of piece will increase tremendously with the number of assets under consideration and the number of period to the end. Development of efficient numerical solution schemes for $(A(\lambda, w))$ seems to be the key in dealing with constrainted investment.

6.2.2 Including Higher Moments

Early in 1970, Samuelson[40] claimed that while the mean-variance formulation is a good approximation to utility functions. Using higher moments may further improve the solutions. The skewness of a distribution is a measurement of symmetricity. A distribution is said to be positvely skewed (skew to right) if it is of a shape like Figure 6.1. On the contrary, if a distribution looks like Figure 6.2, then it is said to be negatively skewed (skew to left). A symmetric distribution is of zero skewness.

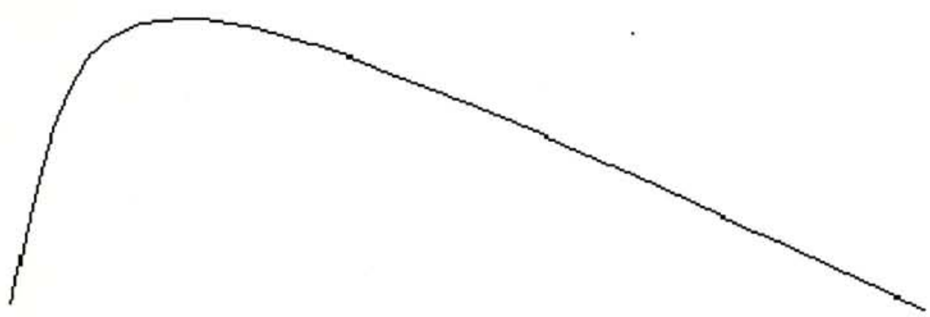

Figure 6.1: Positively Skewed Distribution (Skewed to right)

We now propose a model for solving problem considering not only mean and

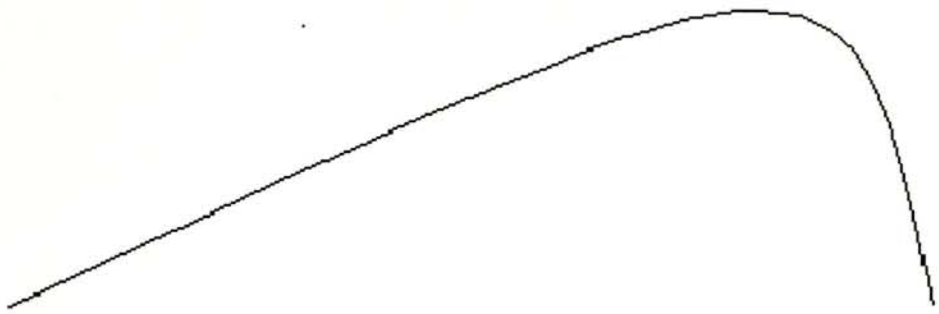

Figure 6.2: Negatively Skewed Distribution (Skewed to left)

variance, but also skewness, i.e. the situations where a utility has the form of $U\left(E\left(x_T\right), Var\left(x_T\right), Skew\left(x_T\right)\right)$. It is reasonable to assume that rational investors will prefer negative skewness to positive one. Hence our objective function could be

$$\begin{aligned} & \max U\left(E\left(x_T\right), Var\left(x_T\right), Skew\left(x_T\right)\right) && (6.4) \\ = \; & \max E\left(x_T\right) - w_1 Var\left(x_T\right) - w_2 Skew\left(x_T\right) && (6.5) \end{aligned}$$

where $w_1, w_2 > 0$

A suggested measurement of Skewness $Skew\left(x_T\right)$ is the third moment,

$$Skew\left(x_T\right) = E\left(\left(x_T - E\left(x_T\right)\right)^3\right) \tag{6.6}$$

It can be further expressed by

$$Skew\left(x_T\right) = E\left(x_T^3\right) - 3E\left(x_T\right)E\left(x_T^2\right) + 2E^3\left(x_T\right) \tag{6.7}$$

Hence we can contruct auxiliary problem in terms of $E\left(x_T^3\right)$,$E\left(x_T^2\right)$and $E\left(x_T\right)$.

$$\begin{aligned}
U &= E\left(x_T\right) - w_1 Var\left(x_T\right) - w_2 Skew\left(x_T\right) \\
&= E\left(x_T\right) - w_1\left(E\left(x_T^2\right) - E^2\left(x_T\right)\right) \\
&\quad - w_2\left(E\left(x_T^3\right) - 3E\left(x_T\right)E\left(x_T^2\right) + 2E^3\left(x_T\right)\right) \\
&= -w_2 E\left(x_T^3\right) + \left(-w_1 + 3w_2 E\left(x_T\right)\right)E\left(x_T^2\right) \\
&\quad + E\left(x_T\right) + w_1 E^2\left(x_T\right) - 2w_2 E^3\left(x_T\right) \\
&= \widetilde{U}\left(E\left(x_T^3\right), E\left(x_T^2\right), E\left(x_T\right)\right)
\end{aligned}$$

where

$$\widetilde{U} = -w_2 E\left(x_T^3\right) + \lambda_1 E\left(x_T^2\right) + \lambda_2 E\left(x_T\right)$$

If we can find out the relationship between the original problem and auxiliary problem. we are able to solve the problem by solving the auxiliary problem. We can then search for the optimal parametric vector $\gamma = [\gamma_1, \gamma_2]'$, where $\gamma_i = \frac{\lambda_i}{w_i}$, in a similar way as we did in the multi-period mean-variance portfolio selection. Thus we can find the optimal solution required to maximaze $U\left(E\left(x_T\right), Var\left(x_T\right), Skew\left(x_T\right)\right)$.

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